1.9.2 Causality and the Feynman propagator

The next topic is causality and its relation to the Feynman propagator. Causality refers to the fact that physical effects cannot travel faster than the speed of light. If we, e.g., in QM compute the transition amplitude $\langle(t',\mathbf{r}')|(t,\mathbf{r})\rangle$ and find a non-zero result for $t'=t$ then it violates causality since physics at $\mathbf{r}$ will affect physics at $\mathbf{r}'$ instantaneously. The forward light-cone is invariant under **continuous Lorentz transformations**, that is all Lorentz transformations that are continuously connected to the identity, i.e., the unit matrix (no boosts and no space rotations). This means that in a Minkowski diagram particles must move in time inside or on the *forward* light-cone.

As will be elaborated upon in connection with the Dirac equation the full Lorentz group contains also the discrete transformations T and P, time and space reversals (parity), respectively. In this context the continuous Lorentz transformations are called the **proper orthochronous Lorentz group** and is denoted $\Lambda_+^\uparrow$ having ${\Lambda^0}_0 > 0$ and $\det\Lambda > 0$.

To put the causality issue into context we analyse it in three cases, non-relativistic QM, relativistic QM, and Klein-Gordon QFT.

1) Non-relativistic QM

Consider the transition amplitude $U(\mathbf{r}_2,\mathbf{r}_1;t)$ for a particle propagating from $\mathbf{r_1}$ to $\mathbf{r_2}$ in time t. This time evolution is generated by $\hat{H}=\frac{\hat{\mathbf{p}}^2}{2m}$ and the amplitude is therefore given by (here objects with hats are operators)

$$U(\mathbf{r}_2,\mathbf{r}_1;t)=\langle\mathbf{r}_2|e^{-it\hat{H}}|\mathbf{r}_1\rangle=\langle\mathbf{r}_2|e^{-it\frac{\hat{\mathbf{p}}^2}{2m}}|\mathbf{r}_1\rangle=\int\frac{d^3p}{(2\pi)^3}e^{-it\frac{\mathbf{p}^2}{2m}}e^{i\mathbf{p}\cdot(\mathbf{r}_2-\mathbf{r}_1)}, \tag{1.228}$$

which is obtained by inserting two complete sets of momentum states and using $\langle\mathbf{p}|\mathbf{r}\rangle = e^{-i\mathbf{p}\cdot\mathbf{r}}$. The momentum integrals are done directly one direction at a time by completing the square and using $\int_{-\infty}^{\infty}dx\,e^{-ax^2}=\sqrt{\frac{\pi}{a}}$. Since the exponent in question is purely imaginary one might add to it a real part $-\epsilon p^2$ (in each direction) and let ϵ go to zero at the end. The result is, in terms of $r := |\mathbf{r}_2-\mathbf{r}_1|$,

$$U(\mathbf{r}_2,\mathbf{r}_1;t)=(\frac{m}{2\pi it})^{3/2}e^{\frac{im}{2t}r^2}, \tag{1.229}$$

which clearly is not causal. The transition probability is given by $|U(\mathbf{r}_2,\mathbf{r}_1;t)|^2$.

2) Relativistic QM

The difference to the previous case is the Hamiltonian which now is $\hat{H}_{rel}=\sqrt{\hat{\mathbf{p}}^2+m^2}$. The transition amplitude then becomes

$$U(\mathbf{r}_2,\mathbf{r}_1;t)=\langle\mathbf{r}_2|e^{-it\hat{H}_{rel}}|\mathbf{r}_1\rangle=\langle\mathbf{r}_2|e^{-it\sqrt{\hat{\mathbf{p}}^2+m^2}}|\mathbf{r}_1\rangle=\int\frac{d^3p}{(2\pi)^3}\,e^{-ip\cdot(x_2-x_1)}, \tag{1.230}$$

where the exponent has become relativistic while the integral is not since it lacks a factor $E_{\mathbf{p}}^{-1}$ as we learned in the previous lecture. On the RHS above $p^0=E_{\mathbf{p}}=\sqrt{\mathbf{p}^2+m^2}$. Writing the exponent as $p\cdot x = tE_{\mathbf{p}} - pr\cos\theta$ (with the somewhat sloppy notation that after the equality sign p means $|\mathbf{p}|$ and $r = |\mathbf{r}_2 - \mathbf{r}_1|$, while $t = t_2 - t_1$). Doing the angle integrals in $\int d^3p = \int_0^\infty dp \int d\Omega$ we get

$$U(\mathbf{r}_2, \mathbf{r}_1; t) = \frac{1}{2\pi^2 r} \int_0^\infty dp\, p \sin(pr)\, e^{-it\sqrt{p^2+m^2}}. \tag{1.231}$$

This integral can either be done exactly and seen to yield a modified Bessel function or analysed by means of the "stationary phase approximation" (or steepest descent method).

Using the latter approach we write $\sin(pr)$ as two exponentials and consider the total exponent in the integrand $f(p) = pr \mp t\sqrt{p^2 + m^2}$. The parts of the p integral where the exponent fluctuates wildly give small contributions to the answer so one can pick up the leading contributions by looking for extremal points of the exponent. So if $\bar{p}$ is an extremum of $f(p)$, i.e., $f'(\bar{p}) = 0$, an expansion around $\bar{p}$ (recall that here $p := |\mathbf{p}|$) gives

$$f(p) = f(\bar{p}) + f'(\bar{p})(p - \bar{p}) + \frac{1}{2} f''(\bar{p})(p - \bar{p})^2 + ..., \tag{1.232}$$

where the first term is a constant and gives an exponential factor outside the integral, the second term is zero and the third one gives the first non-vanishing contribution to the integrand which is then Gaussian (although with an i in the exponent).

To perform these steps using $f(p) = pr \mp t\sqrt{p^2 + m^2}$ we first determine $\bar{p}$:

$$f'(\bar{p}) = r \mp \frac{\bar{p}t}{\sqrt{\bar{p}^2 + m^2}} = 0 \Rightarrow \bar{p} = \pm i \frac{m}{\sqrt{r^2 - t^2}}. \tag{1.233}$$

This result should then be used in $f(\bar{p})$ to get the exponential factor which is

$$U(\mathbf{r}_2, \mathbf{r}_1; t) \propto e^{f(\bar{p})} = e^{\mp m\sqrt{r^2 - t^2}}. \tag{1.234}$$

In addition to this the Gaussian integral gives a factor $\frac{t}{(r^2-t^2)^{5/4}}$. This result can also be obtained from the modified Bessel function in the limit $r^2 - t^2 \to \infty$. Thus we see that for a given time t the correlation has infinite range in r and is therefore not casual (non-zero for $r^2 > t^2$).

3. QFT

In QFT the object that best resembles the transition amplitudes discussed above in QM is, for a real scalar field, (compare to the relativistic amplitude above)

$$D(x_2 - x_1) := \langle 0|\phi(x_2)\phi(x_1)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot(x_2 - x_1)}|_{p^0 = E_{\mathbf{p}}}, \tag{1.235}$$

which follows from

$$\langle 0|a_{\mathbf{p}'} a^\dagger_{\mathbf{p}}|0\rangle = \langle 0|[a_{\mathbf{p}'}, a^\dagger_{\mathbf{p}}]|0\rangle = (2\pi)^3 \delta^3(\mathbf{p}' - \mathbf{p}). \tag{1.236}$$

If we interpret this as the amplitude for a particle created at time t_1 at $\mathbf{r}_1$ to be destroyed at time t_2 at $\mathbf{r}_2$, we can check causality by considering space-like values for $x_2 - x_1$ (i.e.,

$(x_2 - x_1)^2 < 0$ in this metric) and choose a frame where $t_2 - t_1 = 0$. This gives, with $p := |\mathbf{p}|$ and doing the angle integrals,

$$D(x_2 - x_1)|_{(t_2=t_1)} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{i\mathbf{p}\cdot(\mathbf{r}_2-\mathbf{r}_1)} = -\frac{i}{8\pi^2 r} \int_{-\infty}^{\infty} dp\, p \frac{e^{ipr}}{\sqrt{p^2+m^2}}. \tag{1.237}$$

The last integral is a bit messy to do since it requires moving contours around so that they follow the branch cuts of the integrand. Skipping the details, this procedure gives, for large r, the answer

$$D(x_2 - x_1)|_{(t_2=t_1)} \to e^{-mr} \text{ in the limit } r \to \infty. \tag{1.238}$$

This is also obtained, in the same limit, from the exact result which is in terms of a modified Bessel function

$$D(x_2 - x_1)|_{(t_2=t_1)} = \frac{m}{4\pi^2 r} K_1(mr). \tag{1.239}$$

Again we fail to achieve causality but in QFT there is a remedy namely to consider instead the "propagator of effects" given by the commutator of two fields as follows

$$\langle 0|[\phi(x_2), \phi(x_1)]|0\rangle = \langle 0|\phi(x_2)\phi(x_1)|0\rangle - \langle 0|\phi(x_1)\phi(x_2)|0\rangle = D(x_2 - x_1) - D(x_1 - x_2). \tag{1.240}$$

The reason for considering the commutator is that for spatial separation of the two points the two fields must have effects that are independent of each other and hence it must be possible to measure them simultaneously. As above, for spatial separations of the two points we can go the frame where $t_2 = t_1$ and then the commutator becomes an "equal time" commutator which we know is zero from before. In fact, this could also be seen as a consequence of the fact that $D(\mathbf{r}_2 - \mathbf{r}_1)$ is independent of the direction of $\mathbf{r}_2 - \mathbf{r}_1$ as we saw above.

Comment: Recall that, using the mode expansion of the quantum scalar field, we computed previously the equal time commutator in full detail and found, with $\Pi(x) = \dot{\phi}(x)$,

$$[\phi(x_2), \Pi(x_1)] = i\delta^3(\mathbf{r_2} - \mathbf{r_1}). \tag{1.241}$$

Repeating this calculation for two fields or two canonical momenta we get

$$[\phi(x_2), \phi(x_1)] = [\Pi(x_2), \Pi(x_1)] = 0. \tag{1.242}$$

Exercise: Verify that the last two commutators vanish.

Finally we have found an object that displays causality! What is actually going on here is not that obvious in the case of a real scalar field. However, by turning to complex fields it becomes clear that causality in QFT is a consequence of the presence of anti-particles. In fact, in we consider

$$\langle 0|[\Phi(x_2), \Phi^\dagger(x_1)]|0\rangle = \langle 0|\Phi(x_2)\Phi^\dagger(x_1)|0\rangle - \langle 0|\Phi^\dagger(x_1)\Phi(x_2)|0\rangle, \tag{1.243}$$

then the first term on the RHS creates a particle (using $a^\dagger$) with charge q at x_1 and destroys it (using a) at x_2, while the second term does the opposite, namely creates (using $b^\dagger$) an anti-particle (creating $-q$ = destroying q) at x_2 and destroys it (using b to destroy $-q$ i.e., create q) at x_1. Therefore, the two terms in the commutator have the same effect at the two events x_1 and x_2 and cancel each other out for space-like separated events.

Let us now study this causal object to get a better understanding of it. We have

$$\langle 0|[\phi(x_2), \phi(x_1)]|0\rangle = D(x_2-x_1)-D(x_1-x_2) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}(e^{-ip\cdot(x_2-x_1)}-e^{-ip\cdot(x_1-x_2)})|_{p^0=E_{\mathbf{p}}}. \tag{1.244}$$

For $t_2 > t_1$ this can be written as

$$\langle 0|[\phi(x_2), \phi(x_1)]|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}(e^{-ip\cdot(x_2-x_1)}|_{p^0=E_{\mathbf{p}}} - e^{-ip\cdot(x_2-x_1)}|_{p^0=-E_{\mathbf{p}}}), \tag{1.245}$$

where the exponents are now ordered in the same way. Note that in the second term we have defined $p^0 = -E_{\mathbf{p}}$, and to get the whole exponent in the form $(x_2 - x_1)$ we have also flipped the integration variable in this term by letting $\mathbf{p} \to -\mathbf{p}$.

The RHS above can now be written as an integral over p^0 from $-\infty$ to ∞ as follows

$$\langle 0|[\phi(x_2), \phi(x_1)]|0\rangle = \int \frac{d^3p}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{dp^0}{2\pi i}(\frac{-1}{p^2-m^2})e^{-ip\cdot(x_2-x_1)}. \tag{1.246}$$

The reason this expression gives the above result is that for $t_2 > t_1$ we can create a closed contour in the complex p^0 plane by extending the real p^0 axis with a half-circle in the lower half-plane and let its radius tend to ∞. Explicitly, we let $p^0 \to p^0 + iy$. This creates an extra factor since $e^{-i(t_2-t_1)p^0} \to e^{(t_2-t_1)y}\, e^{-i(t_2-t_1)p^0}$ which goes to zero as $y \to -\infty$, i.e., as the radius of half-circle in the lower half-plane tends to infinity. In the upper plane $y \to +\infty$ which means that the half-circle contribution only vanishes for $t_2 < t_1$. Having constructed these closed contours we can then use the theory of residues which gives a non-zero result if there are any simple poles inside the closed contour. Since the half-circles do not contribute for the relations between t_2 and t_1 discussed above, the residue results are equal to the integrals along the real axis of p^0 that we want to compute.

To apply this idea to the p^0 integral above we must first locate the simple poles. This is easily done using the fact that

$$\frac{1}{p^2-m^2} = \frac{1}{2E_{\mathbf{p}}}\left(\frac{1}{p^0-E_{\mathbf{p}}} - \frac{1}{p^0+E_{\mathbf{p}}}\right). \tag{1.247}$$

Thus the poles are located at $p^0 = \pm E_{\mathbf{p}}$ on the real p^0 axis which needs some care.

The trick is to move the poles off the real axis, i.e., we add a small imaginary part $-i\epsilon$ to each pole in such a way that the poles end up below the real axis. Note that for the closed contour in the lower half-plane the integration direction is opposite to the one used in the residue theorem which is the reason for the factor (-1) in the integrand above. This relocation of the poles is done by

$$\frac{1}{p^2-m^2} \to \frac{1}{2E_{\mathbf{p}}}\left(\frac{1}{p^0-E_{\mathbf{p}}+i\epsilon} - \frac{1}{p^0+E_{\mathbf{p}}+i\epsilon}\right). \tag{1.248}$$

Then we compute the result by residues and after that let $\epsilon \to 0$. The result is the one quoted above which is the **retarded propagator** $D_R(x_2 - x_1)$. Here causality is due to the fact that both the poles are in the lower half-plane which gives a non-zero result only for $t_2 > t_1$. Note that, as already mentioned above, the closed contour trick can be used in the upper half-plane only when $t_2 < t_1$. In this case, with both poles below the real axis, the residue vanishes and so does the p^0 integral, in accord with the properties of the retarded propagator.

Brief review of the residue theorem

We will use the residue theorem several times in this course so let us review it here. The residue theorem is applicable to closed paths integrals in the complex plane where the integrand is a holomorphic function (i.e., depends on z but not $\bar{z}$ and has no branch cuts). If the simple poles inside the closed contour C are made explicit the theorem reads

$$\Sigma_i \oint_C dz \frac{f(z)}{z - z_i} = 2\pi i\, \Sigma_i\, f(z_i). \tag{1.249}$$

The contour is in the counterclockwise direction. Expanding the whole holomorphic (=analytic, but often called meromorphic if it includes the poles) function $F(z)$ in the integrand in a Laurent series $F(z) = \Sigma_{n=-\infty}^{\infty} a_n\, (z - z_i)^n$ around each pole gives the residue theorem in terms of the sum of the coefficients a_{-1} for each pole.

To get some feeling for this theorem consider first the trivial integral over an angle θ from $\theta = 0$ to $\theta = 2\pi$: $\int_0^{2\pi} d\theta = 2\pi$. For a circle with radius r this integral gives instead $2\pi r$. Using polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ we can form the complex coordinate $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$. Integrating $\oint dz$ around a circle with fixed r gives then $\oint_r dz = ri \int_0^{2\pi} d\theta e^{i\theta} = 0$. However, if the integrand is $1/z$ we get $\oint \frac{dz}{z} = i \int_0^{2\pi} d\theta = 2\pi i$ which now is independent of the radius of the circle! This indicates that if we subtract this result for two different radii the area between the circles gives no contribution to the integral. In fact, this is true for an area of any shape that does not contain any $1/z$ poles. This follows since an integrand with any other power of z than -1 will always give a zero result. E.g., $\oint \frac{dz}{z^2} = -\oint dz \partial_z(\frac{1}{z}) = 0$ since the integrand is a total derivative in z.

Returning to the retarded propagator $D_R(x_2 - x_1)$ we note that it can be expressed in terms of the step function $\Theta(x)$ as follows

$$D_R(x_2 - x_1) = \Theta(t_2 - t_1)\langle 0|[\phi(x_2), \phi(x_1)]|0\rangle = \Theta(t_2 - t_1)(D(x_2 - x_1) - D(x_1 - x_2)). \tag{1.250}$$

We can now show that $D_R(x_2 - x_1)$ is a Green's function:

$$(\Box_2 + m^2) D_R(x_2 - x_1) = -i\delta^4(x_2 - x_1). \tag{1.251}$$

To check this we compute

$$(\Box_2 + m^2)\Theta(t_2 - t_1)\langle 0|[\phi(x_2), \phi(x_1)]|0\rangle = (\partial_{t_2}^2 \Theta(t_2 - t_1))\langle 0|[\phi(x_2), \phi(x_1)]|0\rangle$$

$$+2(\partial_{t_2}\Theta(t_2 - t_1))\langle 0|[\dot{\phi}(x_2), \phi(x_1)]|0\rangle + \Theta(t_2 - t_1)\langle 0|[(\Box_2 + m^2)\phi(x_2), \phi(x_1)]|0\rangle. \tag{1.252}$$

While the last term vanishes we have to deal with the first two terms. They can be simplified by using the relation between a step function and a delta function, namely $\partial_x \Theta(x) = \delta(x)$: the first term on the RHS above then contains a factor $\partial_{t_2}\delta(t_2 - t_1))$ and the second one a factor $\delta(t_2 - t_1))$.

Finally, $\partial_{t_2}\delta(t_2 - t_1)$ is not a nice object but fortunately it can be simplified by making use of the fact that the equal time commutator of two scalar fields vanishes. This fact can be expressed as

$$\delta(t_2 - t_1)\langle 0|[\phi(x_2), \phi(x_1)]|0\rangle = 0. \tag{1.253}$$

If we hit this equation with a t_2 derivative we find that the troublesome term becomes

$$(\partial_{t_2}\delta(t_2 - t_1))\langle 0|[\phi(x_2), \phi(x_1)]|0\rangle = -\delta(t_2 - t_1)\langle 0|[\dot{\phi}(x_2), \phi(x_1)]|0\rangle, \tag{1.254}$$

which thus combines with the middle term above giving the final result (recall that $\dot{\phi} = \Pi$)

$$(\Box_2 + m^2)D_R(x_2 - x_1) = \delta(t_2 - t_1)\langle 0|[\Pi(x_2), \phi(x_1)]|0\rangle = -i\delta^4(x_2 - x_1). \tag{1.255}$$

But Green's functions can be computed by Fourier transformation:

$$G(x_2 - x_1) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x_2 - x_1)}\tilde{G}(p). \tag{1.256}$$

The defining equation for the Green's function then immediately implies

$$(\Box_2 + m^2)G(x_2 - x_1) = -i\delta^4(x_2 - x_1) \;\Rightarrow\; \tilde{G}(p) = \frac{i}{p^2 - m^2}, \tag{1.257}$$

which hence means that

$$G(x_2 - x_1) = \int \frac{d^4p}{(2\pi)^4}\, \frac{i}{p^2 - m^2} e^{-ip\cdot(x_2 - x_1)}. \tag{1.258}$$

Is this the retarded Green's function? Yes and no! It can be any of four possible Green's functions depending on how the poles are located relative the real p^0-axis when computing the p^0 integral. In fact, at this point it is up to us to define where the poles are and thus which Green's function we want to obtain. We can do what we did above, namely to shift both poles below the real axis and get the retarded Green's function. Shifting them above the real axis gives instead the so called advanced Green's function.

The amazing fact discovered by Feynman is that the most useful Green's function, at least in relativistic QFT, is obtained if we shift the pole at $p^0 = -E_{\mathbf{p}}$ above the real axis and the pole at $p^0 = +E_{\mathbf{p}}$ below the real axis! This leads to the **Feynman propagator**. (The fourth possibility does not seem be of any interest.) It is a nice feature of this propagator that this particular way of moving the poles off the real axis is generated by the following insertion of $i\epsilon$:

$$\tilde{G}(p) = \frac{i}{p^2 - m^2} \to \tilde{G}(p) = \frac{i}{p^2 - m^2 + i\epsilon}. \tag{1.259}$$

The locations of the poles are then: $p^0 = \pm\sqrt{(E_{\mathbf{p}}^2 - i\epsilon)} \approx \pm E_{\mathbf{p}}(1 - \frac{i\epsilon}{2E_{\mathbf{p}}}) = \pm E_{\mathbf{p}} \mp \frac{i}{2}\epsilon$.

Comment: This course does not contain any material on the very important subject of path integrals. But there is one interesting aspect that is closely connected to the pole structure of the Feynman propagator and thus might be a suitable comment to make here.

The Feynman path integral is mathematically a very complicated object since it involves integrating over spaces of functions. This is usually written as

$$Z \propto \int \mathcal{D}\phi \, e^{\frac{i}{\hbar}S[\phi]}, \tag{1.260}$$

where Z is called the partition function and $S[\phi]$ is the action functional for some **classical field** here denoted ϕ. This expression for Z can either be used as a generating function for the Feynman rules which is a very efficient way to derive them, or one can actually try to compute it somehow. One way to make sense of it is to perform a Wick rotation (which will later play a very important role also in this course), that is, turning time imaginary by the replacement $t \to -it$.

Performing the Wick rotation $t \to -it$ in the path integral gives the exponent

$$\begin{aligned}
&i\int dt \int d^3r \frac{1}{2}(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2) \to i\int(-idt)\int d^3r\frac{1}{2}(-\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2) \\
&= -\int dt \int d^3r\frac{1}{2}(\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2) = -\frac{1}{2}\int d^4x(\phi(x)\,(-\Box_{eucl} + m^2)\phi(x)) \\
&= -\frac{1}{2}\int d^4p\tilde{\phi}(p)\,(p^2 + m^2)\tilde{\phi}(-p),
\end{aligned} \tag{1.261}$$

where in the last step we have performed a Fourier transformation to Euclidean momentum space which implies $\Box_{eucl} \to -p^2$ which is now a Euclidean sum of terms.

The Wick rotation in time used here can be related to a Wick rotation in p^0. This latter one is defined by the closed contour obtained by rotating the real p^0 axis in the opposite direction to the time rotation, i.e., $p^0 \to +ip^0$, which is possible since this rotation will NOT pass over any of the two Feynman poles in the p^0 plane.

There are two nice properties emerging here:
1. The above expression for the integrand is negative definite since $p^2 + m^2$ is positive definite in Euclidean signature.
2. Since this is the exponent in the partition function Z it has become a infinite dimensional Gaussian integral like $\int dx\, e^{-x^2}$ and can thus be computed by using the mode expansions and turning it into an integral over the expansion coefficients.
For Hawking and his collaborators the Euclidean path integral was a key tool and in some sense was defined to be the starting point for physics instead of the Lorentz invariant formulation we are used to. It has important applications, e.g, in describing tunneling phenomena in terms of instantons.

Time ordering: Returning to the Feynman propagator we now want to introduce the concept of time ordering which will play an key role in the development of perturbation theory later.

The Feynman propagator derived above can be written

$$D_F(x_2 - x_1) = \begin{cases} +D(x_2 - x_1), & \text{for } t_2 > t_1, \\ +D(x_1 - x_2), & \text{for } t_2 < t_1, \end{cases}$$

$$= \Theta(t_2 - t_1)\langle 0|\phi(x_2)\phi(x_1)|0\rangle + \Theta(t_1 - t_2)\langle 0|\phi(x_1)\phi(x_2)|0\rangle, \tag{1.262}$$

where the plus sign arises because the contour in the upper half-plane emerges with the correct orientation. This result defines *time ordering* which is denoted by a capital T:

$$D_F(x_2 - x_1) := \langle 0|T(\phi(x_2)\phi(x_1))|0\rangle, \tag{1.263}$$

and is a prescription telling us to sum over all possible time ordered terms with later operators to the left of earlier ones (here we have only two ϕ fields but it can involve any number of operators of different kinds as we will see later).

Meaning of a quantum field? Classical bosonic, i.e., integer spin, fields like A_μ in EM and by analogy scalar fields Φ are in principle measurable quantities. This is not the case for the quantised versions of these fields since they are operators. Therefore one should evaluate them on specific states and compute their matrix expectation values to have a change to interpret them. Doing this we could try to compare the results to ordinary QM where we probably understand them better. So let us consider a real scalar quantum field at $t = 0$ for simplicity. Then the mode expansion implies

$$\phi(\mathbf{r})|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-i\mathbf{p}\cdot\mathbf{r}}|\mathbf{p}\rangle, \tag{1.264}$$

and by taking the hermitian conjugate of this result we find that

$$\langle 0|\phi(\mathbf{r})|\mathbf{p}\rangle = e^{-i\mathbf{p}\cdot\mathbf{r}}. \tag{1.265}$$

Some feeling for the quantum ϕ is then offered by the comparison to similar objects in QM:

$$|\mathbf{r}\rangle = \int \frac{d^3p}{(2\pi)^3}\, e^{-i\mathbf{p}\cdot\mathbf{r}}|\mathbf{p}\rangle, \tag{1.266}$$

and

$$\langle \mathbf{r}|\mathbf{p}\rangle = e^{-i\mathbf{p}\cdot\mathbf{r}}. \tag{1.267}$$

Causality? Finally, we must return to the issue of causality. The important point here is that the Feynman propagator does not respect causality as the retarded one $D_R(x_2 - x_1)$ does. Therefore causality must checked again at some later point in the development of perturbation theory. In Weinberg, Vol. 1, 145 this is done by considering the Hamiltonian density $\mathcal{H}$ and the behaviour of the following commutator at space-like distances,

$$[\mathcal{H}(x_2), \mathcal{H}(x_1)] = 0, \text{ for } (x_2 - x_1)^2 < 0. \tag{1.268}$$

This condition can, in fact, be verified and thus causality established despite the fact that the formalism is based on Hamiltonians and therefore does not display causality in any obvious way.