1.10 Dirac theory

In this chapter (Chap 3 in PS) we will apply the construction of Lorentz covariant quantum fields, developed above for scalar fields, to particles with charge and spin, in this case $s = 1/2$. Before doing that we need, however, to derive the Dirac equation for ψ, the spinorial Dirac field, and its Lagrangian. The concept of a Dirac spinor will be explained in the process. These two ways to approach Dirac theory is in some sense complementary and may provide a better understanding of it.

The aspects of Dirac theory discussed after this derivation of the theory concern
1. Dirac matrices and their properties,
2. free particle solutions,
3. the quantum Dirac field and the Dirac propagator, and
4. discrete symmetries and the PCT theorem.

1.10.1 The Dirac equation

Below we will obtain the Dirac equation by imposing Lorentz invariance on ordinary two-component wave functions in the quantum mechanical description of electrons. This is not to be viewed as a strict derivation but rather as trick that will provide the answer you are aiming at.

Thus the starting point here is the two-component wave function χ used for spin 1/2 particles in non-relativistic QM. Recall first the Hamiltonian for a scalar particle:

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m}. \tag{1.269}$$

For spin 1/2 particles, the Hamiltonian acts on two-component complex wave functions χ, referred to as **spinors** from now on. Hence $\hat{H}$ should be expressed in terms of Pauli matrices σ^i as follows:

$$\hat{H} = \frac{(\hat{\mathbf{p}} \cdot \sigma)^2}{2m}. \tag{1.270}$$

The Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.271}$$

and satisfy

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}, \ [\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k, \tag{1.272}$$

which can be summarised in the single (very useful) matrix equation

$$\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k. \tag{1.273}$$

The fact that the two versions of the Hamiltonian given above (i.e., with and without Pauli matrices) are equal follows directly from the fact that two momenta commute with each other:

$$(\hat{\mathbf{p}} \cdot \sigma)(\hat{\mathbf{p}} \cdot \sigma) = \hat{p}^i \hat{p}^j \sigma^i \sigma^j = \hat{\mathbf{p}}^2. \tag{1.274}$$

This last fact will survive if we couple the particle (now having a charge) to a vector potential $\mathbf{A}(x)$, as is familiar from QM. Then

$$\hat{H} = \frac{((\hat{\mathbf{p}} - e\mathbf{A}) \cdot \sigma)^2}{2m}, \tag{1.275}$$

with drastic consequences if we try to get rid of the Pauli matrices. In particular, we have to be careful about the fact that $\hat{\mathbf{p}}$ and $\mathbf{A}(x)$ don't commute. We get

$$\hat{H} = \frac{((\hat{\mathbf{p}} - e\mathbf{A}) \cdot \sigma)^2}{2m} = \frac{(\hat{p}^i - eA^i)(\hat{p}^j - eA^j)\sigma^i\sigma^j}{2m} = \frac{1}{2m}(\hat{p}^i\hat{p}^j - e\hat{p}^iA^j - eA^i\hat{p}^j + e^2A^iA^j)\sigma^i\sigma^j. \tag{1.276}$$

Since the momenta are effectively derivatives we should write the second term as $\hat{p}^iA^j = (\hat{p}^iA^j) + A^j\hat{p}^i$ where the bracket indicates that the momentum operators act only the vector potential and not further to the right. Using this in the Hamiltonian gives

$$\hat{H} = \frac{1}{2m}(\hat{p}^i\hat{p}^j - e(\hat{p}^iA^j) - e(A^i\hat{p}^j + A^j\hat{p}^i) + e^2A^iA^j)\sigma^i\sigma^j. \tag{1.277}$$

Then using the product role for the Pauli matrices above we see that this Hamiltonian can be written

$$\hat{H} = \frac{(\hat{\mathbf{p}} - e\mathbf{A})^2}{2m} - \frac{ie}{2m}\epsilon^{ijk}(\hat{p}^iA^j)\sigma^k. \tag{1.278}$$

Exercise: Verify the last expression for the Hamiltonian.
The second term is very interesting since it is related to the coupling of a magnetic field to the magnetic moment of the particle. This follows from using $\hat{p}^i = -i\partial_i$, since then

$$\frac{ie}{2m}\epsilon^{ijk}(\hat{p}^iA^j)\sigma^k = \frac{e}{2m}\epsilon^{ijk}(\partial^iA^j)\sigma^k = \frac{e}{2m}\mathbf{B} \cdot \sigma = \frac{e}{m}\mathbf{B} \cdot \mathbf{S} := \frac{ge}{2m}\mathbf{B} \cdot \mathbf{S}, \tag{1.279}$$

where $\mathbf{S}$ is the spin operator. This implies that the Lande's g-factor is determined to $g = 2$ just from the facts that the particle has charge e and spin 1/2. We will have ample reasons to return to this $g = 2$ result later since this is one of the best measured quantities in QED, see David Gross at Solvay 2011[29].

Imposing Lorentz invariance on the spin 1/2 particle above leads immediately to the Dirac equation. To understand how this works we first consider a scalar particle in QM and write the relativistic equation $E^2 - \mathbf{p}^2 = m^2$ in terms of the derivative representation $\hat{p}_\mu = (\hat{E}, -\hat{\mathbf{p}}) = i\partial_\mu$ and let it act on a scalar wave function denoted ϕ. We then get $\hat{E}^2 - \hat{\mathbf{p}}^2 = -(\partial_t^2 - \nabla^2) = -\Box$ and hence

$$-(\hat{E}^2 - \hat{\mathbf{p}}^2) + m^2 = 0 \Rightarrow (\Box + m^2)\phi(x) = 0, \tag{1.280}$$

i.e., the Klein-Gordon equation. This is not a proper derivation of the Klein-Gordon equation since it requires a reinterpretation of the function ϕ, from being first a wave function in QM to becoming a scalar field in QFT. In this sense the procedure is more of an educated trick than a derivation to obtain the answer one is looking for.

[29]For a recent, March 2020, discussion of these issues, see Jana et al, ArXiv hep-ph/2003.03386. You should be able to read the first two pages.

Repeating this in the spin 1/2 case requires again Pauli matrices and gives, now acting on two-component spinors, denoted as ψ from now on to conform to standard practise:

$$(E - \mathbf{p}\cdot\sigma)(E + \mathbf{p}\cdot\sigma) = m^2 \;\Rightarrow (i\partial_t + i\nabla\cdot\sigma)(i\partial_t - i\nabla\cdot\sigma)\psi = m^2\psi. \tag{1.281}$$

The last step is to write this second order differential equation on a two-component spinor as a first order equation on a four-component spinor. To do this we rename ψ as ψ_L and define $\psi_R := \frac{1}{m}(i\partial_t - i\nabla\cdot\sigma)\psi_L$ giving two first order (in derivatives) equations

$$(i\partial_t - i\sigma\cdot\nabla)\psi_L = m\psi_R, \tag{1.282}$$

$$(i\partial_t + i\sigma\cdot\nabla)\psi_R = m\psi_L. \tag{1.283}$$

These equations together constitute the Dirac equation. Its standard form is obtained by defining the **four-component spinor** ψ and the four four-dimensional Dirac matrices γ^μ as follows (below we use the time-space split $\mu = (0, i)$)

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \; \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \text{ where } \sigma^\mu = (\mathbf{1}, \sigma^i),\; \bar{\sigma}^\mu = (\mathbf{1}, -\sigma^i). \tag{1.284}$$

The two first order equations can then be rewritten as a single equation, the Dirac equation, which reads

$$(i\gamma^\mu\partial_\mu - m)\psi = 0. \tag{1.285}$$

Since constructed from the Pauli matrices it is natural to expect that the Dirac matrices also anti-commute nicely. In fact

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{1.286}$$

Note, however, that the commutation relations $[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$ have no direct analogue for the Dirac matrices. As mentioned earlier the commutator and anti-commutator relations satisfied by the Pauli matrices can be summarised as

$$\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k. \tag{1.287}$$

In the case of the Dirac matrices the corresponding relation reads

$$\gamma^\mu\gamma^\nu = g^{\mu\nu} + \gamma^{\mu\nu}, \text{ where } \gamma^{\mu\nu} := \gamma^{[\mu}\gamma^{\nu]} := \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu). \tag{1.288}$$

There is a large number of other very important identities for the γ^μ matrices that we will discuss later.

One useful fact about the γ matrices that we may comment on immediate is that there are infinitely many choices of four 4×4 matrices that satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. They are, however, all *equivalent*. This notion of equivalence means that different versions can be related to each other by similarity transformations, i.e., that an S exists such that $\gamma'^\mu = S\gamma S^{-1}$. In particular, if we take the hermitian conjugate, the complex conjugate or the transpose of the γ^μs we get new γ-matrices satisfying the same anti-commutation relations. Therefore, there exist three matrices A, B and C such that

$$(\gamma^\mu)^\dagger = \pm A\gamma^\mu A^{-1},\ (\gamma^\mu)^\star = \pm B\gamma^\mu B^{-1},\ (\gamma^\mu)^T = \pm C\gamma^\mu C^{-1}. \tag{1.289}$$

For instance, using the γ-matrices defined previously we find that in the first relation above $A = \gamma^0$ if the plus sign is chosen. Note that $(\gamma^\mu)^\dagger = \pm\gamma^\mu$ with plus sign for $\mu = 0$ and minus sign for $\mu = i$ which is in fact true in any version of the matrices. The matrices A and C will be used below.

There are two very important special cases of Dirac's theory, obtained by restricting the Dirac spinor to either Weyl spinors or Majorana spinors. Both have important applications in particle physics as well as in condensed matter physics.

Weyl spinors: If we set the mass m in the above derivation of the Dirac equation to zero the two complex spinors ψ_L and ψ_R decouple from each other resulting in two two-component Lorentz covariant (see below) equations

$$i\sigma^\mu\partial_\mu\psi_R = 0,\ i\bar{\sigma}^\mu\partial_\mu\psi_L = 0. \tag{1.290}$$

In the standard model, before the Higgs effect has given all the spin 1/2 fields masses, the whole Lagrangian is expressed entirely in terms of this kind of two-component Weyl spinors. In fact, as we will discuss in more detail later, ψ_L and ψ_R must not appear symmetrically in a Lorentz invariant Lagrangian which opens up the possibility to have a theory that is not parity invariant (i.e., the theory and its mirror image might not be equivalent). As it turns out, Nature is not parity invariant. Neutrinos are also, even after the Higgs effect has occurred, described by Weyl spinors but today we know that these particles have very tiny masses so the Weyl spinor description must be extended somehow. It will be clear below that ψ_L and ψ_R are two inequivalent irreps of the Lie algebra of the Lorentz group[30].

Majorana spinors: Just as for scalar fields we may also in the Dirac case start from the complex field ψ for particles with charge and restrict it to be real which only describes neutral particles. The condition that provides this restriction is called the **Majorana condition** and reads, in terms of the four-component complex Dirac spinors introduced above,

$$\bar{\psi} = \tilde{\psi},\ \text{where}\ \bar{\psi} := \psi^\dagger\gamma^0,\ \text{and}\ \tilde{\psi} := \psi^T C. \tag{1.291}$$

Here $\bar{\psi}$ is the Dirac conjugate and $\tilde{\psi}$ the Majorana conjugate. The comment made above that there are many different explicit versions of the gamma matrices here becomes very useful. If it would be case that the matrix $C = \gamma^0$ then the Majorana condition clearly tells us that all the four components of ψ are real. This happens in the so called **Majorana representation** of the gamma matrices. However, for the gamma matrices we are using, the chiral ones, the Majorana condition still reduces the eight real function in the Dirac

[30]This statement is related to the fact that the Lorentz algebra $so(1,3)$ is isomorphic to $sl(2,\mathbf{C})$.

spinors to four but they are distributed differently, namely they end up in one complex two-component spinor which then give rise to both ψ_L and ψ_R. In our chiral representation of the gamma matrices C will satisfy the above condition with the minus sign. Then, as is easily checked recalling that γ^μ is symmetric for $\mu = 0, 2$ and antisymmetric for $\mu = 1, 3$,

$$C\gamma^\mu C^{-1} = -(\gamma^\mu)^T \;\Rightarrow\; C = i\gamma^0\gamma^2. \tag{1.292}$$

With this particular C matrix the Majorana condition reads

$$\psi^\dagger\gamma^0 = i\psi^T\gamma^0\gamma^2 \;\Rightarrow\; \psi^\dagger = -i\psi^T\gamma^2. \tag{1.293}$$

Using the explicit form of the chiral gamma matrices gives

$$C = i\gamma^0\gamma^2 = i\begin{pmatrix} 0 & \sigma^0 \\ \bar{\sigma}^0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ \bar{\sigma}^2 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}. \tag{1.294}$$

Finally using $i\sigma^2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) := \epsilon$ we see that

$$C = \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \tag{1.295}$$

Then inserting $\psi^T = (\psi_L^T, \psi_R^T)$ and similarly for $\psi^\dagger$, into the Majorna condition it becomes

$$\psi^\dagger\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = \psi^T\begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \;\Rightarrow\; (\psi_R^\dagger, \psi_L^\dagger) = (-\psi_L^T\epsilon, \psi_R^T\epsilon) \;\Rightarrow\; \psi_R = -\epsilon^T\psi_L^\star = \epsilon\psi_L^\star. \tag{1.296}$$

Note that the two conditions on the two-component spinors are identical. The result of these manipulations is often expressed by renaming the upper two-component spinor ψ_L as χ. Then the Majorana spinor ψ_M reads

$$\psi_M = \begin{pmatrix} \chi \\ \epsilon\chi^\star \end{pmatrix}. \tag{1.297}$$

Majorana-Weyl spinors: Trying to impose both conditions simultaneously does not work. This will set the spinor to zero. The above discussion about Weyl and Majorana spinors is very dependent on the dimension (and signature) of spacetime. In e.g. ten-dimensional Minkowski space Majorana-Weyl spinors do exist which is crucial for many field theories (supergravity) and the superstring.

The Lorentz group and its Lie algebra: We have already seen that the $so(3)$ Lie algebra is isomorphic to $su(2)$ and reads

$$[T^i, T^j] = i\epsilon^{ijk}T^k. \tag{1.298}$$

It has the following two representations heavily used in physics, the vector T and spinor S^i irreps:

$$(T^i)_{jk} = -i\epsilon_{ijk}, \;\; S^i = \frac{1}{2}\sigma^i. \tag{1.299}$$

To generalise the vector irrep to higher dimensions one must renumber the generators T^i so that the indices refer to the plane the rotation is carried out in: $T^1 := T^{23}$ etc. This means that in three dimensions

$$T^{ij} = \epsilon^{ijk} T^k \;\; \Rightarrow \;\; (T^{ij})_{kl} = -2i\delta^{ij}_{kl} := -i(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k). \tag{1.300}$$

The $so(3)$, or any $so(N)$, algebra then reads

$$[T^{ij}, T^{kl}] = i\left((-\delta^{jk})T^{il} + ...\right), \tag{1.301}$$

where ... refers to the three extra terms needed to make RHS antisymmetric in both ij and kl. The generalisation to the non-compact Lorentz algebra is then straightforward: let $-\delta^{ij}$ be the space part of the Lorentz metric in PS: $g^{\mu\nu} = diag(1, -\delta^{ij})$. Then the Lorentz algebra is, denoting the generators in a general irrep as J (as in PS),

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left((g^{\nu\rho})J^{\mu\sigma} + ...\right). \tag{1.302}$$

Explicitly the spin 1 generators $T^{\mu\nu}$ and the spin 1/2 generators $S^{\mu\nu}$ are

$$(T^{\mu\nu})_{\rho\sigma} = 2i\delta^{\mu\nu}_{\rho\sigma}, \tag{1.303}$$

and

$$S^{\mu\nu} = \frac{i}{2}\gamma^{\mu\nu}. \tag{1.304}$$

Exercise: Check that the explicit matrices $(T^{\mu\nu})_{\rho\sigma} = 2i\delta^{\mu\nu}_{\rho\sigma}$ satisfy the Lorentz algebra above.

It is very useful to be a bit more explicit about the spin 1/2 generators. Thus

$$S^{\mu\nu} = \frac{i}{2}\begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix}, \text{ where } \sigma^{\mu\nu} := \sigma^{[\mu}\bar{\sigma}^{\nu]}, \;\; \bar{\sigma}^{\mu\nu} := \bar{\sigma}^{[\mu}\sigma^{\nu]}. \tag{1.305}$$

and hence (using the time-space split $\mu = (0, i)$)

$$S^{0i} = \frac{i}{2}\begin{pmatrix} \sigma^{0i} & 0 \\ 0 & \bar{\sigma}^{0i} \end{pmatrix} = \frac{i}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{1.306}$$

$$S^{ij} = \frac{i}{2}\begin{pmatrix} \sigma^{ij} & 0 \\ 0 & \bar{\sigma}^{ij} \end{pmatrix} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \tag{1.307}$$

where we have used the fact that both σ^{ij} and $\bar{\sigma}^{ij}$ are equal to $-i\epsilon^{ijk}\sigma^k$. Now we also see why the upper two components ψ_L and lower two ones ψ_R constitute two different, in fact inequivalent (not obvious), irreps of the Lorentz algbra, and why this algebra is isomorphic to $sl(2, \mathbf{C})$ ($S^{\mu\nu}$ is block diagonal).

There is another extremely important fact to learn from the above explicit forms of S^{0i} and S^{ij}: while S^{0i} are anti-hermitian the S^{ij} are hermitian. The generators are used to construct elements of the Lorentz group $SO(1, 3)$ or $SL(2, \mathbf{C})$ by exponentiation

$$\Lambda = e^{-\frac{i}{2}\epsilon_{\mu\nu}J^{\mu\nu}}. \tag{1.308}$$

Thus we conclude that the spin 1/2 irrep is not unitary, which is also true for the spin 1 generators above. This is, in fact, true for all finite-dimensional irreps (and thus reps) of the Lorentz group as a direct consequence of the unitarity theorem stated previously.

Note that the minus sign in the exponent of Λ has the nice consequence that in the spin 1 case (in a somewhat sloppy notation but the last one is OK)

$$(\Lambda)_{\rho\sigma} = (e^{-\frac{i}{2}\epsilon_{\mu\nu}J^{\mu\nu}})_{\rho\sigma} = e^{-\frac{i}{2}\epsilon_{\mu\nu}2i\delta^{\mu\nu}_{\rho\sigma}} = e^{\epsilon_{\rho\sigma}} = (e^{\epsilon})_{\rho\sigma}, \tag{1.309}$$

which looks a bit better with one index up and one down, i.e., (but still sloppy)

$$(\Lambda)^{\rho}{}_{\sigma} = e^{\epsilon^{\rho}{}_{\sigma}} = (e^{\epsilon})^{\rho}{}_{\sigma}. \tag{1.310}$$

Our next task will be to show that the Dirac equation is Lorentz invariant. This fact should be clear already from our derivation but showing it involves some useful computations that we will do here. The first thing to show is that the Dirac matrices are numerically invariant matrices just as the Pauli matrices. Thus we should show that the Lorentz transformation of the Dirac matrices gives back exactly the same matrices, whose matrices indices are here denoted $a, b, ..,$:

$$(\gamma^{\mu})_a{}^b \rightarrow (\gamma'^{\mu})_a{}^b := \Lambda^{\mu}{}_{\nu}(\Lambda_a{}^c(\gamma^{\nu})_c{}^d\Lambda^{-1b}_{d}). \tag{1.311}$$

Here we are keeping track of the spin irrep of the Lorentz transformation by their matrix indices, $\mu, \nu, ..$ for vectors and $a, b, ...$ for (four-component) spinors. Thus we must use for spin 1

$$(\Lambda)^{\rho}{}_{\sigma} = (e^{-\frac{i}{2}\epsilon_{\mu\nu}T^{\mu\nu}})^{\rho}{}_{\sigma}, \tag{1.312}$$

and for spin 1/2

$$(\Lambda)_a{}^b = (e^{-\frac{i}{2}\epsilon_{\mu\nu}S^{\mu\nu}})_a{}^b. \tag{1.313}$$

To first order in the parameters ϵ we get

$$(\gamma'^{\mu})_a{}^b = (\gamma^{\mu})_a{}^b - \frac{i}{2}\epsilon_{\rho\sigma}\left((T^{\rho\sigma})^{\mu}{}_{\nu}(\gamma^{\nu})_a{}^b + (S^{\rho\sigma})_a{}^c(\gamma^{\mu})_c{}^b - (\gamma^{\mu})_a{}^c(S^{\rho\sigma})_c{}^b\right). \tag{1.314}$$

Inserting the explicit expressions for the generators this becomes

$$(\gamma'^{\mu})_a{}^b = (\gamma^{\mu})_a{}^b + \epsilon^{\mu}{}_{\nu}(\gamma^{\nu})_a{}^b + \frac{1}{4}\epsilon_{\rho\sigma}(\gamma^{\rho\sigma}\gamma^{\mu} - \gamma^{\mu}\gamma^{\rho\sigma})_a{}^b. \tag{1.315}$$

Exercise: Show that the last expression follows from the previous one.

Thus we need to compute the commutator $[\gamma^{\rho\sigma}, \gamma^{\mu}]$. A nice method would be to give a complete list of matrices that span the space of 4×4 matrices. We will do this later in our more systematic study of the Dirac matrices but for now we concentrate on this commutator. We use instead a more standard method which is OK here but is not very nice in more complicated cases. First consider (without displaying the spinor indices)

$$[\gamma^{\rho}\gamma^{\sigma}, \gamma^{\mu}] = \gamma^{\rho}\{\gamma^{\sigma}, \gamma^{\mu}\} - \{\gamma^{\rho}, \gamma^{\mu}\}\gamma^{\sigma} = 2\gamma^{\rho}g^{\sigma\mu} - 2g^{\mu\rho}\gamma^{\sigma}. \tag{1.316}$$

Antisymmetrising this over ρ and σ gives, since $\gamma^{[\rho}\gamma^{\sigma]} := \gamma^{\rho\sigma}$,

$$[\gamma^{\rho\sigma}, \gamma^{\mu}] = 2\gamma^{[\rho}g^{\sigma]\mu} - 2g^{\mu[\rho}\gamma^{\sigma]} = 4\gamma^{[\rho}g^{\sigma]\mu}, \tag{1.317}$$

implying directly that the Dirac matrices are invariant:

$$\gamma'^{\mu} = \gamma^{\mu} + \epsilon^{\mu}{}_{\nu}\gamma^{\nu} + \epsilon_{\rho\sigma}\gamma^{[\rho}g^{\sigma]\mu} = \gamma^{\mu} + \epsilon^{\mu}{}_{\nu}\gamma^{\nu} + \epsilon_{\rho}{}^{\mu}\gamma^{\rho} = \gamma^{\mu}, \tag{1.318}$$

as we set out to prove. Note the steps used in $\epsilon_{\rho\sigma}\gamma^{[\rho}g^{\sigma]\mu} = \epsilon_{\rho\sigma}\gamma^{\rho}g^{\sigma\mu} = \epsilon_{\rho}{}^{\mu}\gamma^{\rho}$.

Lorentz invariance of the Dirac equation

When proving that the Klein-Gordon equation is Lorentz invariant we start from the defining property of a Lorentz transformation, namely that it leaves the bilinear form $x^2 := g_{\mu\nu}x^{\mu}x^{\nu}$ invariant. Thus, using $x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu}$, this condition implies

$$x'^2 = g_{\mu\nu}x'^{\mu}x'^{\nu} = g_{\mu\nu}\Lambda^{\mu}{}_{\rho}x^{\rho}\Lambda^{\nu}{}_{\sigma}x^{\sigma} = x^2. \tag{1.319}$$

The invariance condition in the last equality then implies

$$g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}, \tag{1.320}$$

which means that the Lorentz group can be defined as the set of matrices $\Lambda^{\mu}{}_{\nu}$ for which the Minkowski metric is an invariant tensor (i.e., it stays numerically the same).

It is then easy to show that the Klein-Gordon equation is invariant also. This follows from the fact that[31]

$$dx'^{\mu} = \Lambda^{\mu}{}_{\nu}dx^{\nu} \;\Rightarrow\; \partial'_{\mu} = \partial_{\nu}(\Lambda^{-1})^{\nu}{}_{\mu}. \tag{1.321}$$

Then the invariance of the Klein-Gordon equation follows from $g^{\mu\nu}$ being an invariant matrix under Lorentz transformations and that scalar fields transform as

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x). \tag{1.322}$$

This may look unfamiliar but if we first rename x as x' and then define $x := \Lambda^{-1}x'$, it becomes

$$\phi'(x') = \phi(x). \tag{1.323}$$

The reason one introduces the previous form of the transformation is that it can be viewed as an effect of an **active** coordinate transformation, e.g., physically moving an object. Consider a fixed coordinate system in space and a particle located at $\mathbf{r} = 0$. A function that can describe this situation is the delta-function $\delta(\mathbf{r})$. If an active transformation is used to move the particle to $\mathbf{r} = \mathbf{a}$ the delta-function must become $\delta(\mathbf{r} - \mathbf{a})$, i.e., be given by an inverse translation: ($\hat{\mathbf{p}} = -i\nabla$)

$$T_{(-\mathbf{a})}\delta(\mathbf{r}) = e^{-i\mathbf{a}\cdot\hat{\mathbf{p}}}\delta(\mathbf{r}) = e^{-\mathbf{a}\cdot\nabla}\delta(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{a}). \tag{1.324}$$

[31] In matrix notation these transformations read $dx' = dx\Lambda^T$ and $\partial' = (\Lambda^{-1})^T\partial$ which immediately implies the identity $dx'^{\mu}\partial'_{\mu} = dx^{\mu}\partial_{\mu}$.

This we write in general as

$$T_{(\mathbf{a})} \Rightarrow \phi'(x) = \phi(T^{-1}_{(\mathbf{a})}x). \tag{1.325}$$

For vector and spinor fields the above discussion leads to the rules (compare to ∂_μ)

$$A_\mu(x) \to A'_\mu(x) = (\Lambda^{-1})^\nu{}_\mu A_\nu(\Lambda^{-1}x), \text{ or } A'_\mu(x') = (\Lambda^{-1})^\nu{}_\mu A_\nu(x), \tag{1.326}$$

and

$$\psi(x) \to \psi'(x) = \Lambda_{(1/2)}\psi(\Lambda^{-1}x) \text{ or } \psi'(x') = \Lambda_{(1/2)}\psi(x). \tag{1.327}$$

Then the Dirac operator (note that γ^μ is not transformed at this stage)

$$\gamma^\mu\partial_\mu\psi(x) \to \gamma^\mu\partial'_\mu\psi'(x') = \gamma^\mu\partial_\nu(\Lambda^{-1}_{(1)})^\nu{}_\mu\Lambda_{(1/2)}\psi(x). \tag{1.328}$$

But having proven above that the Dirac matrices are invariant objects under Lorentz transformations, i.e. $\gamma^\mu = \Lambda^\mu_{(1)}{}_\nu(\Lambda_{(1/2)}\gamma^\nu\Lambda^{-1}_{(1/2)})$, we see directly that the Dirac operator is covariant and hence that the Dirac equation is invariant.
Exercise: Show that the statements in the last sentence is correct.

1.10.2 The Dirac Lagrangian

The Lagrangian that gives the Dirac equation as its Euler-Lagrange (EL) equation is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi, \text{ where } \bar{\psi} := \psi^\dagger\gamma^0. \tag{1.329}$$

In Lagrangians built from complex fields like ψ the field variation $\delta\psi$ and $\delta\bar{\psi}$ are regarded as independent. Thus a $\delta\bar{\psi}$ variation leads directly to the Dirac equation. In fact, the EL equations reads here

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} = (i\gamma^\mu\partial_\mu - m)\psi = 0, \tag{1.330}$$

since $\bar{\psi}$ appears in $\mathcal{L}$ only without derivatives acting on it. The Dirac equation follows also from a $\delta\psi$ variation but then in its hermitian conjugated form (see below).

There are a couple of important point to learn from this Lagrangian. First we should prove that the definition $\bar{\psi} := \psi^\dagger\gamma^0$ is a consequence of Lorentz invariance. An infinitesimal Lorentz transformation on the spinor fields with parameters ϵ gives

$$\delta_\epsilon\mathcal{L} = \delta_\epsilon\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\delta_\epsilon\psi. \tag{1.331}$$

Here

$$\delta_\epsilon\psi = -\frac{i}{2}(\frac{i}{2}\epsilon_{\mu\nu}\gamma^{\mu\nu})\psi = \frac{1}{4}\epsilon_{\mu\nu}\gamma^{\mu\nu}\psi \Rightarrow \delta_\epsilon\psi^\dagger = \frac{1}{4}\epsilon_{\mu\nu}\psi^\dagger(\gamma^{\mu\nu})^\dagger, \tag{1.332}$$

which when multiplied from the right by γ^0 and inserting $(\gamma^0)^2 = 1$ becomes

$$\delta_\epsilon\bar{\psi} = \delta_\epsilon\psi^\dagger\gamma^0 = \frac{1}{4}\epsilon_{\mu\nu}\psi^\dagger(\gamma^{\mu\nu})^\dagger\gamma^0 = \frac{1}{4}\epsilon_{\mu\nu}\bar{\psi}\gamma^0(\gamma^{\mu\nu})^\dagger\gamma^0. \tag{1.333}$$

To calculate $\gamma^0(\gamma^{\mu\nu})^\dagger\gamma^0$ we need just to use the $\dagger$-identity discussion above, which for $A = \gamma^0 = (\gamma^0)^{-1}$ reads $\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu$. This implies directly that

$$\gamma^0(\gamma^{\mu\nu})^\dagger\gamma^0 = \gamma^0(\gamma^{[\nu})^\dagger(\gamma^{\mu]})^\dagger\gamma^0 = \gamma^{\nu\mu} = -\gamma^{\mu\nu}. \tag{1.334}$$

It is enough to consider the mass term and hence we see that the two terms in the variation cancels implying Lorentz invariance. Clearly without the γ^0 matrix in the definition of $\bar{\psi}$ this would not have worked.

This result is also connected to the issue of unitarity of the Lorentz transformation used here which are for the finite-dimensional spinor representation of ψ. $\psi^\dagger\psi$ is a product that is preserved under unitary transformations so the need to insert γ^0 between the ψs implies that the Lorentz transformations are not generated by unitary matrices, which we anyway know from before is not possible.

Finally, the Lagrangian must be a real quantity (after quantisation it must be hermitian as an operator). Checking this for the mass term goes as follows

$$(\bar{\psi}\psi)^\dagger = (\psi^\dagger\gamma^0\psi)^\dagger = \psi^\dagger(\gamma^0)^\dagger\psi = \bar{\psi}\psi. \tag{1.335}$$

Repeating this for the kinetic term:

$$(\bar{\psi}(i\gamma^\mu\partial_\mu)\psi)^\dagger = -i(\partial_\mu\psi^\dagger)(\gamma^\mu)^\dagger(\gamma^0)^\dagger\psi = -i(\partial_\mu\bar{\psi})\gamma^\mu\psi = \bar{\psi}(i\gamma^\mu\partial_\mu)\psi, \tag{1.336}$$

where the last equality is the result of an integration by parts since the whole calculation is supposed to be done under the integral $\int d^4x$.

Comment: Note that the kinetic term does not mix ψ_L and ψ_R since $\gamma^0\gamma^\mu$ is 2×2 block diagonal. The mass term on the other hand does mix these two Weyl spinors since γ^0 is block off-diagonal. This fact will be important later.

Comment: If we restrict the terms in the Lagrangian to Majorana spinors we see that the mass term becomes

$$\bar{\psi}\psi = \psi^T C\psi, \tag{1.337}$$

which is only non-zero if the spinor fields anti-commute: using index notation ($a = 1, 2, 3, 4$)

$$\psi_a\psi_b = -\psi_b\psi_a. \tag{1.338}$$

1.10.3 Mode expansions and the free-particle solutions of the Dirac equation

In the same spirit as we constructed the scalar field mode expansions before we now address the mode expansion of the Dirac field ψ. The first step is to list the creation and annihilation operators needed to deal with all the different kinds of physical degrees of freedom in this case: the particles have spin 1/2 so the particle can be in either a spin up or a spin down state. Together with charges $\pm q$ and momentum $\mathbf{p}$ we therefore need the operators (we don't to start from tilde operators this time since all the steps are the same as for the scalar fields)

$$Q = +q: \;\; a^s_{\mathbf{p}},\; a^{s\dagger}_{\mathbf{p}},\; Q = -q: \;\; b^s_{\mathbf{p}},\; b^{s\dagger}_{\mathbf{p}}. \tag{1.339}$$

The spin states are labelled by the upper index $s = (up, down) = \pm\frac{1}{2}$ (i.e., s is not spin but rather the magnetic quantum number usually denoted m, see below). The mode expansion must therefore read

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \Sigma_s \left(a_{\mathbf{p}}^s u^s(p) e^{-ip\cdot x} + b_{\mathbf{p}}^{s\dagger} v^s(p) e^{ip\cdot x} \right), \tag{1.340}$$

where the 4-component spinors $u^s(p)$ and $v^s(p)$ have been introduced since ψ itself is such a spinor. They play a role similar to the polarisation tensor for vector fields. But how are they determined? We saw that the mode expansion for the scalar fields satisfied the Klein-Gordon equation so here we should impose the Dirac equation.

Concentrating first on one particular mode, $u^s(p)e^{-ip\cdot x}$, the Dirac equation is

$$(p\cdot\gamma - m)u^s(p) = 0 \Rightarrow \begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} u^s(p) = 0. \tag{1.341}$$

To get a basic understanding of this equation we first solve it in the rest frame where $p^\mu = (m, 0, 0, 0)$:

$$\begin{pmatrix} -m & m \\ m & -m \end{pmatrix} u^s(m) = 0 \Rightarrow u^s(m) = \sqrt{m} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix}, \tag{1.342}$$

where the factor $\sqrt{m}$ is for later convenience. The two two-component spinors ξ^s for $s = \pm\frac{1}{2} = (1, 2)$ must span this two-dimensional space so we take them to be

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \;\; \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \xi^{s\dagger}\cdot\xi^{s'} = \delta^{ss'}. \tag{1.343}$$

These two spinors ξ^s can be associated with spin up and spin down since they satisfy

$$S^3\xi^1 = +\frac{1}{2}\xi^1, \;\; S^3\xi^2 = -\frac{1}{2}\xi^2, \text{ where } S^3 = S^{12} = \frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}. \tag{1.344}$$

In a general Lorentz frame where the momentum is $p^\mu \neq (m, 0, 0, 0)$ the solution for $u^s(p)$ is (note the $\bar{\sigma}^\mu$)

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix}. \tag{1.345}$$

This can be proven by performing a Lorentz boost on $u^s(m)$ from $p^\mu = (m, 0, 0, 0)$ to a general $p^\mu \neq (m, 0, 0, 0)$ (as done in PS), or by checking that $u^s(p)$ solves the Dirac equation. The latter is easier and instructive so let's do that:

$$\begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} = 0. \tag{1.346}$$

The top two components go as follows:

$$-m\sqrt{p\cdot\sigma}\,\xi^s + p\cdot\sigma\sqrt{p\cdot\bar{\sigma}}\,\xi^s = -m\sqrt{p\cdot\sigma}\,\xi^s + \sqrt{p\cdot\sigma}\sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})}\,\xi^s = 0, \tag{1.347}$$

where we have used $(p\cdot\sigma)(p\cdot\bar{\sigma}) = (p^0 - p^i\sigma^i)(p^0 + p^i\sigma^i) = (p^0)^2 - p^ip^j\sigma^i\sigma^j = m^2$ in the last equality. The last two components work the same way.

At this point it is possible to explain the notation ψ_L and ψ_R, which refer to left and right. For massless spin 1/2 particles we can discuss their helicity. Consider such particles moving in the positive z-direction and quantise the spin, as above (since eigenstates of S^3), in the same direction. Then ψ_L satisfies the Weyl equation

$$\bar{\sigma}^\mu\partial_\mu\psi_L = (\sigma^0\partial_t - \sigma^i\partial_i)\psi_L = 0. \tag{1.348}$$

Using plane waves in the z direction $e^{\pm i(p^0t - p^3z)}$ where $p^\mu = (E,0,0,E)$ the Weyl equation becomes

$$E(\mathbf{1}+\sigma^3)\psi_L = 0 \Rightarrow \sigma^3\psi_L = -\psi_L \Rightarrow h := \hat{\mathbf{p}}\cdot\mathbf{S} = -\frac{1}{2}, \tag{1.349}$$

that is, ψ_L describes particles with helicity $h = -\frac{1}{2}$, i.e., left-handed ones.

For the coming calculations of scattering amplitudes we should now derive a number of very useful relations for the $u^s(p)$ and $v^s(p)$ spinors. They are of two kinds: scalar products and completeness relations similar to the ones in QM: $\langle\mathbf{p}'|\mathbf{p}\rangle = (2\pi)^3\delta^3(\mathbf{p}'-\mathbf{p})$ and $\int\frac{d^3p}{(2\pi)^3}|\mathbf{p}\rangle\langle\mathbf{p}| = \mathbf{1}$, respectively.

The derivations use the square root trick above and give the following scalar products (note: same p^μ, different spin directions s, p^μ and σ^μ are hermitian)

$$u^{s\dagger}(p)u^{s'}(p) = (\xi^{s\dagger}\sqrt{p\cdot\sigma}, \xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}})\begin{pmatrix}\sqrt{p\cdot\sigma}\xi^{s'}\\ \sqrt{p\cdot\bar{\sigma}}\xi^{s'}\end{pmatrix} = \xi^{s\dagger}(p\cdot\sigma)\xi^{s'} + \xi^{s\dagger}(p\cdot\bar{\sigma})\xi^{s'}$$

$$= \xi^{s\dagger}(p^0 - p^i\sigma^i)\xi^{s'} + \xi^{s\dagger}(p^0 + p^i\sigma^i)\xi^{s'} = 2E_{\mathbf{p}}\delta^{ss'}. \tag{1.350}$$

In a similar way, by inserting a γ^0, we get a Lorentz invariant result

$$\bar{u}^s(p)u^{s'}(p) = 2m\delta^{ss'}. \tag{1.351}$$

These calculations for $u^s(p)$ can be done also for the other spinor in ψ, that is, $v^s(p)$. Since the mode function in this case has the opposite sign in the exponent compared to the case for $u^s(p)$ the Dirac equation gives the same result as for $u^s(p)$ apart from a very important minus sign

$$v^s(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\xi^s\\ -\sqrt{p\cdot\bar{\sigma}}\xi^s\end{pmatrix}. \tag{1.352}$$

We have here chosen to expand in the same two-dimensional basis ξ^s as for the $u^s(p)$ spinors which will lead to a question about the spin direction later. These spinors satisfy the following scalar product relations, again with an extremely important sign difference,

$$v^{s\dagger}(p)v^{s'}(p) = 2E_{\mathbf{p}}\delta^{ss'}, \tag{1.353}$$

and, with an extra γ^0,

$$\bar{v}^s(p)v^{s'}(p) = -2m\delta^{ss'}. \tag{1.354}$$

One also finds the mixed scalar product relations

$$\bar{u}^s(p)v^{s'}(p) = 0, \tag{1.355}$$

but

$$u^{s\dagger}(p)v^{s'}(p) = 2\xi^s(p^i\sigma^i)\xi^{s'} \neq 0, \tag{1.356}$$

while however

$$u^{s\dagger}(p)v^{s'}(\bar{p}) = \bar{u}^s(p)v^{s'}(p) = 0, \tag{1.357}$$

where we have defined $\bar{p}^\mu = (p^0, -\mathbf{p})$.

Turning to the completeness relations, here called **spin sums**, we get

$$\Sigma_s u^s(p)\bar{u}^s(p) = \Sigma_s \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix} (\xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}}, \xi^{s\dagger}\sqrt{p\cdot\sigma}), \tag{1.358}$$

where we should use the completeness relation $\Sigma_s\, \xi^s\xi^{s\dagger} = \binom{1}{0}(1,0) + \binom{0}{1}(0,1) = \mathbf{1}$. Doing that the above spin sum becomes

$$\Sigma_s u^s(p)\bar{u}^s(p) = \begin{pmatrix} \sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})} & p\cdot\sigma \\ p\cdot\bar{\sigma} & \sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)} \end{pmatrix} = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix}, \tag{1.359}$$

which simply means that

$$\Sigma_s u^s(p)\bar{u}^s(p) = \gamma\cdot p + m. \tag{1.360}$$

In a similar way we get for the $v^s(p)$ spinors, again with an important sign flip,

$$\Sigma_s v^s(p)\bar{v}^s(p) = \gamma\cdot p - m. \tag{1.361}$$

1.10.4 Chiral currents and some gamma-ology

We are now going to discuss chiral currents which play an important role in the construction of the standard model in particle physics. Before doing this we will, however, first develop a more systematic picture of the gamma matrices that will be useful also in the discussion of discrete symmetries and the CPT theorem at the end of this chapter.

Recall the structure of the chiral form of the gamma matrices used in this course

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \text{ where } \sigma^\mu = (1, \sigma^i),\ \bar{\sigma}^\mu = (1, -\sigma^i). \tag{1.362}$$

Just as we can span the set of all 2×2 matrices by $(\mathbf{1}, \sigma^i)$ we would like to construct a basis for the space of all 4×4 matrices using the gamma matrices γ^μ. This is done by constructing 16 linearly independent matrices as follows:

$$\mathbf{1}, \gamma^\mu, \gamma^{\mu\nu}, \gamma^{\mu\nu\rho}, \gamma^{\mu\nu\rho\sigma}, \tag{1.363}$$

which stops at this points since the indices take only four values $\mu = 0,1,2,3$ and gives zero if antisymmetrised over five or more indices: $\gamma^{\mu_1\cdots\mu_n} := \gamma^{[\mu_1}.....\gamma^{\mu_n]}$.

Does the above list constitute a complete basis spanning the space of 4×4 matrices? To answer this question we must first check that there are exactly 16 matrices and then prove that they are all independent. The number of matrices is OK since the list gives $1+4+6+4+1 = 16 = 4 \times 4$ matrices.

Independence is a bit more involved to prove. Before we do this we introduce a somewhat simpler form of the last two cases in the list. The standard definitions are connected to the fact that $\gamma^{\mu\nu\rho\sigma}$ must be proportional to $\epsilon^{\mu\nu\rho\sigma}$ defined here by $\epsilon^{0123} = +1$. Using this fact we define the γ^5 matrix as follows:

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3 = i\begin{pmatrix} \bar{\sigma}^1\sigma^2\bar{\sigma}^3 & 0 \\ 0 & \sigma^1\bar{\sigma}^2\sigma^3 \end{pmatrix} = i\begin{pmatrix} \sigma^1\sigma^2\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^2\sigma^3 \end{pmatrix} = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}. \quad (1.364)$$

Thus we can use γ^5 instead of $\gamma^{\mu\nu\rho\sigma}$ in the complete list of matrices above. In a similar spirit we can replace the four matrices $\gamma^{\mu\nu\rho}$ by $\gamma^\mu\gamma^5$. Note that the position up or down for the 5 does not matter: $\gamma^5 := \gamma_5$. An immediate consequence of its definition is that γ^5 anticommutes with all the γ^μ matrices, i.e., $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ which can be written

$$\{\gamma^\mu, \gamma^5\} = 0. \quad (1.365)$$

We can now establish the fact that all of the basis gamma-matrices except the unit matrix are traceless. First, obviously $tr\gamma^5 = 0$ but this we can prove also by using that $(\gamma^0)^2 = 1$ as follows:

$$tr\gamma^5 = tr(\gamma^5(\gamma^0)^2) = -tr(\gamma^0\gamma^5\gamma^0) = -tr\gamma^5, \quad (1.366)$$

where we used $(\gamma^0)^2 = 1$ in the first step, that $\{\gamma^\mu, \gamma^5\} = 0$ in the second and in the third step that the trace is cyclic. The fact the γ^μ matrices are traceless are also obvious in the chiral representation we use here but this is then true in any representation since the similarity transformations cancel out in the trace. Using the method above we can consider $tr\gamma^\mu(\gamma^5)^2$ to find that γ^μ are traceless. This way one can easily prove that all matrices in the list except the unit matrix are traceless.

The final step in the independence proof is to compute traces like $tr(\gamma^\mu\gamma^{\nu\rho})$. For this one needs to expand the matrix product $\gamma^\mu\gamma^{\nu\rho}$ as a sum of elements in the basis. In this case, and in all other ones, this works as follows: recall the formula $\gamma^\mu\gamma^\nu = g^{\mu\nu} + \gamma^{\mu\nu}$ discussed before. This formula tells us that the two matrices on the LHS have either the same index and then squares to +1 or -1, or have different indices and then belongs to the basis element with two antisymmetric indices. The coefficients in such expansions are always ± 1. For our latest example this means that the expansion reads

$$\gamma^\mu\gamma^{\nu\rho} = \gamma^{\mu\nu\rho} + 2g^{\mu[\nu}\gamma^{\rho]}. \quad (1.367)$$

Note that the factor 2 is needed to get only ± 1 coefficients when writing out the antisymmetrisation explicitly. This equation proves that the trace of the LHS is zero and hence that the basis elements with one and two indices are linearly independent. This can be done generally with any choices of basis elements which proves the fact that the list is a complete basis.

Chiral currents:
With the definition of the γ^5 matrix we can now discuss chiral currents. Let us start from the familiar case of EM coupled to charged spin 1/2 particles whose kinetic term is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - eA_\mu \bar{\psi}\gamma^\mu \psi := \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - A_\mu j^\mu, \quad (1.368)$$

where, as usual, the last equality defines the current $j^\mu = e\bar{\psi}\gamma^\mu\psi$.

By Noether's theorem this current comes from the phase invariance $\psi' = e^{i\alpha(x)}\psi$ and is thus conserved, $\partial_\mu j^\mu = 0$, on-shell. To show this explicitly we need the equations of motion derived from the $\delta\bar{\psi}$ variation, i.e., $\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0$,:

$$(i\gamma^\mu\partial_\mu - m)\psi - eA_\mu\gamma^\mu\psi = 0 \Rightarrow \gamma^\mu\partial_\mu\psi = -im\psi - ieA_\mu\gamma^\mu\psi. \quad (1.369)$$

We also need the Dirac conjugated Dirac equation, so consider first its hermitian conjugate

$$\partial_\mu\psi^\dagger(\gamma^\mu)^\dagger = im\psi^\dagger + ieA_\mu\psi^\dagger(\gamma^\mu)^\dagger, \quad (1.370)$$

which after multiplication from the right by γ^0 to get the Dirac conjugate $\bar{\psi}$ gives

$$\partial_\mu\bar{\psi}\gamma^0(\gamma^\mu)^\dagger\gamma^0 = im\bar{\psi} + ieA_\mu\bar{\psi}\gamma^0(\gamma^\mu)^\dagger\gamma^0. \quad (1.371)$$

Using the identity $\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu$ this equation becomes

$$\partial_\mu\bar{\psi}\gamma^\mu = im\bar{\psi} + ieA_\mu\bar{\psi}\gamma^\mu. \quad (1.372)$$

Then

$$\partial_\mu j^\mu = e\partial_\mu(\bar{\psi}\gamma^\mu\psi) = e(\partial_\mu\bar{\psi})\gamma^\mu\psi + e\bar{\psi}\gamma^\mu\partial_\mu\psi = 0. \quad (1.373)$$

It is now possible to define a new current denoted j_5^μ involving the γ^5 matrix

$$j_5^\mu := \bar{\psi}\gamma^\mu\gamma^5\psi. \quad (1.374)$$

Repeating the above check of on-shell conservation we find that this current is not conserved in general since

$$\partial_\mu j_5^\mu = 2im\bar{\psi}\psi. \quad (1.375)$$

The reason this fails if $m \neq 0$ is that the phase transformation associated to this current is

$$\psi' = e^{i\alpha\gamma^5}\psi \Rightarrow (\bar{\psi}\psi)' \neq \bar{\psi}\psi. \quad (1.376)$$

Thus the mass term in the Lagrangian in not invariant under these new γ^5 phase transformation which of course implies that j_5^μ is not conserved if the mass term is present in the Lagrangian.

Comment: The fact that j_5^μ is not conserved if the Lagrangian contains this mass term does not imply a problem for unitarity simply because there is no gauge field that couples to j_5^μ in the Lagrangian discussed here. In the standard model such gauge fields are present for physics reasons and hence no fermionic mass terms are allowed. All fermionic mass terms must then arise from the Higgs mechanism which is one of the basic principles used in the construction of the standard model. In the standard model this is implemented using the chiral currents defined below.

The standard way to deal with the two currents, j^μ and j_5^μ, is to consider the following two linear combinations, called **chiral currents**, defined by

$$j_L^\mu := \bar{\psi}\gamma^\mu P_L \psi = \psi_L^\dagger \bar{\sigma}^\mu \psi_L, \text{ where } P_L := \frac{1}{2}(1-\gamma^5) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.377}$$

and

$$j_R^\mu := \bar{\psi}\gamma^\mu P_R \psi = \psi_R^\dagger \sigma^\mu \psi_R,, \text{ where } P_R := \frac{1}{2}(1+\gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}. \tag{1.378}$$

Note that these currents don't mix the left and right handed Weyl spinors due to the γ^0 in the Dirac conjugate. At this point it is rather clear that one can introduce gauge fields for both these currents and provided there are no fermionic mass terms, as is the case of the standard model prior to the Higgs effect taking place. Without mass terms both currents are conserved and there are no problems with unitarity (both A_0 fields are then possible to gauge away!).

Comment: In the modern context of the **swampland/landscape** discussion there are arguments from black hole physics that strongly suggest that all continuous symmetries must be gauged. This is expressed in terms of the so called **weak gravity conjecture** which says that gravity is the weakest of all forces. This seems like a true and trivial statement but it implies that one cannot turn off the coupling constant in a gauge theory since eventually the force will become weaker than gravity. String theory is the only theory we know that implies that all symmetries are gauged. The standard model as we know it today does have in it global symmetries, i.e., symmetries that are not gauged.

Comment: The basic standard model is build from massless spin 1/2 fields which get massive due to the Higgs effect. This means that chiral Weyl spinors must team up to form Dirac spinors which contain both ψ_L and ψ_R. This happens for all spin 1/2 fields (electron, quarks etc) but not for the neutrinos. Today we know that neutrinos have very tiny masses (at least for two of the three generations) since the sum of the three types of neutrino masses must not exceed about 1 eV. It is a very curious fact that the energy scale associated with the neutrino masses is very close to the one associated with the cosmological constant. It is not yet experimentally established which extra states Nature has chosen to use for the neutrinos to get non-zero masses. One idea, the so called *see-saw mechanism*, makes use of Majorana spinors.