1.3 Group theory 1: The classical matrix groups

Consider the **set of all** complex $N \times N$ matrices U satisfying the unitarity condition

$$U^\dagger = U^{-1}, \quad \text{or } U^\dagger U = UU^\dagger = \mathbf{1}, \tag{1.5}$$

leaving the scalar product in $\mathbf{C}^N$ invariant: If $\chi \in \mathbf{C}^N$ and $\chi' = U\chi$ then $\chi'^\dagger \chi' = \chi^\dagger \chi$. The unitarity property survives matrix multiplication which means that if two matrices U_1 and U_2 are unitary so is the product $U_3 = U_1 U_2$ which hence also belongs to this matrix set:

$$U_3^\dagger U_3 = (U_1 U_2)^\dagger U_1 U_2 = U_2^\dagger U_1^\dagger U_1 U_2 = \mathbf{1}. \tag{1.6}$$

Note that the unit matrix $\mathbf{1}$ is in this set of matrices and that every matrix has an inverse, which is actually part of the assumption above (since $U^\dagger = U^{-1}$ implies that $\det U = e^{i\alpha}$ with $\alpha \in \mathbf{R}$). We may also add the requirement that the matrices have unit determinant which is also preserved by matrix multiplication since $\det(U_1 U_2) = (\det U_1)(\det U_2)$. This is a subset of the previous one without the unit determinant condition.

If we consider **all** matrices in either one of these two sets the following properties are trivially satisfied:
1. The set is **closed** *under multiplication (here matrix multiplication)*
2. The multiplication is **associative** *(as matrix multiplication always is)*
3. There is a **unit** *(here the unit matrix* $\mathbf{1}$*)*
4. Every element in the set has an **inverse** *(true here since we consider only matrices satisfying* $U^\dagger = U^{-1}$*, i.e., matrices with non-zero determinant*

Viewing these properties instead as axioms they define a **group**, often denoted G and members of the set are called *elements*, which in the case above is called the *unitary* group:

$$U(N), \tag{1.7}$$

realised in this discussion in terms of complex $N \times N$ matrices. When the condition that the matrices have unit determinant $\det U = \mathbf{1}$ is added the group is called *special unitary*:

$$SU(N). \tag{1.8}$$

The standard model of elementary particles is based on three such groups: $U(1)$, $SU(2)$ and $SU(3)$. The group $U(1)$ is the group of *phases*, i.e., multiplication by $e^{i\alpha}$ where α is a (real) parameter (angle), and is *abelian*. The other two groups are *non-abelian*, i.e., $g_1 g_2 \neq g_2 g_1$ for some $g_1, g_2 \in G$. Any $U \in SU(2)$ can be written in terms of $a, b \in \mathbf{C}$ as

$$U \in SU(2): \quad U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \text{ where } |a|^2 + |b|^2 = 1, \tag{1.9}$$

which is easily checked. Two general such matrices do not commute and the group is therefore *non-abelian*. The set of matrices in this group can be parametrised by the points on the unit three-sphere S^3 since $|a|^2+|b|^2 = 1$ which, if written in real variables $a = x_1+ix_2$ and $b = x_3 + ix_4$, becomes $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$, i.e., the equation for S^3 embedded in $\mathbf{R}^4$.

While $U(1)$ and $SU(2)$ can be viewed in this way in terms of simple geometries S^1 (unit complex numbers) and S^3 (unit quaternions)[3] $SU(3)$ and higher $SU(N)$ groups are much more complicated as manifolds and do not correspond to anything with other names.

These notions can be carried over to real matrices leading to the *orthogonal* groups

$$O(N) \text{ or } SO(N). \tag{1.10}$$

Among the groups called *classical* there is just one other case, the *symplectic* groups. One way to define matrix groups is by looking for matrices preserving some special matrices numerically. One example is the unit matrix which then leads to orthogonal groups ($g\mathbf{1}g^T = \mathbf{1} \Rightarrow g \in O(N)$) while invariance of the antisymmetric $2N \times 2N$ matrix

$$C = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}, \tag{1.11}$$

defines the *symplectic* groups[4]

$$Sp(2N). \tag{1.12}$$

Comments:

1. The complete classification of all *finite-dimensional Lie groups* (i.e., continuous of the kind discussed above) is known as the Cartan classification and contains in addition to the *classical* groups above, i.e., the unitary (denoted A_n by Cartan), orthogonal (denoted B_n or D_n), and symplectic (denoted C_n), also the *exceptional* ones G_2, F_4, E_6, E_7, E_8. These latter ones have, however, no simple definition in terms of matrices similar to the one used above. The index n (and the indices appearing on the exceptional groups) is the **rank** of the group related to the maximal number of matrices that can be diagonalised simultaneously. The classes A_n contain also the groups $GL(N)$ and $SL(N)$ which are general matrices with non-zero or unit determinant, respectively.
2. There are also other important groups like *finite* ones with a finite number of group elements, and those with a *discrete* set of elements which is *infinite* in number (see courses in Group theory and in String theory).
3. Other important Lie groups in physics have *infinite dimension.* Examples of such are *Virasoro* and *Kac-Moody* appearing in two-dimensional *conformal field theory (CFT_2)* used, e.g., in string theory and in the context of phase transitions in condensed matter systems.
4. There is a very important (also for QFT) distinction between compact groups (e.g., $U(N), SU(N), SO(N)$) and non-compact ones (e.g., $SO(1,3), SU(1,1), Sp(2N), SL(N)$) discussed further in "Group theory 2" on Lie algebras and representations.

[3]There is a third case, the octonions, for which a unit octonion can be shown to be the same as the seven-dimensional sphere, S^7. This manifold is, however, not a group manifold but does nevertheless share some properties with group manifolds as, e.g., being parallelisable. S^7 plays a key role in string/M theory.

[4]Note that some authors call these groups $Sp(N)$.