1.4 Group theory 2: Lie algebras and representations

1.4.1 Lie algebras: the compact case

Finite groups, like the symmetry group of a cube, have a finite number of group elements and are characterised by the multiplication table of the group elements. For continuous groups, i.e., *Lie groups*, such a table is not meaningful. Instead one turns to the *Lie algebra* associated to the Lie group, which is a local property of the Lie group. Adding information about the global, or topological, properties of the Lie group manifold one has a full description of the group. The standard example is the Lie algebras $so(3)$ and $su(2)$ which are the same (i.e., isomorphic) while their groups $SO(3)$ and $SU(2)$ are locally the same but topologically different: $SU(2) = S^3$ and $SO(3) = RP^3$ which is the set of all lines through the center of the three-sphere S^3. This relation is written $SO(3) = SU(2)/\mathbf{Z}_2$. We will in general denote groups with upper case and Lie algebras with lower case letters.

The Lie algebra can be extracted from a Lie group G by writing its elements g as the exponential of a linear combination of a new set of matrices, the so called *generators* T^i. That is

$$g \in G: \quad g := e^{i\alpha^i T^i}, \;\; i = 1, 2, ..., d, \tag{1.13}$$

with the exponential of a matrix defined by its power series expansion. Here d is the *dimension* of the group or the Lie algebra, and one should note that the parameters, or "angles", α^i are real numbers. The generators can be shown to satisfy commutation relations, known as the *Lie algebra*, of the form

$$[T^i, T^j] = i f^{ij}{}_k T^k, \tag{1.14}$$

where the constants $f^{ij}{}_k$ are called the *structure constants*. Some well-known examples will appear below.[5]

The following relations will be very useful: If M and A are matrices then

$$M := e^A \implies \det M = e^{Tr\,A}, \;\; M^\dagger = (e^A)^\dagger = e^{A^\dagger}. \tag{1.15}$$

Defining $SU(N)$ as the set of all $N \times N$ complex unitary matrices with unit determinant and $SO(N)$ as the set of all $N \times N$ real orthogonal matrices we find, with the imaginary unit i in the exponent as above[6], the following conditions on the generators

$$SU(N): \quad (T^i)^\dagger = T^i, \;\; Tr\, T^i = 0,$$

[5] The generators span a vector space which means that the structure constants depend on which linear combinations of the generators are considered independent. Therefore it is quite tricky to find out which Lie algebra is actually represented by a given set of structure constants. This is solved by using Dynkin diagrams (see courses on group theory and Lie algebras).

[6] The definition of the generators with an i in the exponent is particularly useful in quantum mechanics since it leads to hermitian generators which have real eigenvalues, a property required by *observables* in quantum mechanics. However, sometimes it is instead useful to work with definitions without the i in the exponent, e.g., in purely geometrical considerations. Note that without the i in the exponent the Lie algebra is also written without the i on the right hand side.

$$SO(N): \quad (T^i)^\dagger = T^i, \;\; Tr\, T^i = 0, \tag{1.16}$$

which in the latter case means that the generators are anti-symmetric and purely imaginary since g is a real matrix in this case. The generators so defined are in the *fundamental*, or defining, representation of the group or its Lie algebra. Other *representations*, that is other sets of matrices, satisfying a specific Lie algebra are discussed below. All possible matrix representations[7] of the Lie algebras classified by Cartan are known. E.g., in the case of $su(2)$, or equivalently $so(3)$, we know from QM that these correspond to all integer and half-integer spins j with matrices of dimension $(2j+1) \times (2j+1)$. Thus the spin 1/2 representation is in terms of 2×2 Pauli matrices, acting on a two-dimensional complex vector space of wave functions denoted χ_α with $\alpha = 1, 2$ for the two spin states *up* and *down*.

For $su(2)$ the above conditions on the generators translate into the facts that $i = 1, 2, 3$ and that the generators T^i can be related to the Pauli matrices σ^i:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \;\; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \;\; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.17}$$

which span the space of 2 by 2 complex hermitian traceless matrices and satisfy

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}, \;\; [\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k. \tag{1.18}$$

It is convenient to define

$$T^i = \frac{1}{2}\sigma^i, \tag{1.19}$$

so that the generators satisfy the **Lie algebra** of $SU(2)$, i.e.,

$$su(2) := Lie(SU(2)): \quad [T^i, T^j] = i\epsilon^{ijk}T^k. \tag{1.20}$$

Note that the above anti-commutation relations are a property of the Pauli matrices and not a general feature related to the Lie algebra.

For $su(3)$ the eight corresponding 3×3 matrices can be found in Problem 15.1 in Peskin and Schroeder. They are a factor of $\frac{1}{2}$ times the so called *Gell-Mann matrices* often denoted λ^i. Note that the Pauli matrices appear as 2×2 submatrices in the first three of the Gell-Mann matrices which therefore satisfy the same algebra as $su(2)$: $su(2)$ is thus a *subalgebra* of $su(3)$.

The $su(3)$ generators in Problem 15.1 satisfy $[T^i, T^j] = if^{ij}{}_k T^k$ where the structure constants $f^{ij}{}_k$ are rather complicated. For $su(2)$, on the other hand,

$$su(2): \quad f^{ij}{}_k = \epsilon^{ijk}, \tag{1.21}$$

while for most other Lie algebras, including $su(3)$, there is no such easy way to describe the structure constants.

[7] These are matrices of finite size.

For a general $SU(N)$ the number of generators, i.e., the dimension of the group or its Lie algebra, is easily obtained: The number of imaginary antisymmetric matrices are $\frac{1}{2}N(N-1)$ and the number of real symmetric and traceless ones is $\frac{1}{2}N(N+1)-1$. For $SU(N)$ this gives N^2-1 while for $SO(N)$ we find $\frac{1}{2}N(N-1)$.

Let us now consider the group $SO(3)$. Its generators are real antisymmetric 3×3 matrices[8] multiplied by i. There are exactly three such matrices which can be written

$$(T^i)_{jk} = -i\epsilon^{ijk} \quad i.e. \;\; T^1 = -i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \;\; etc, \tag{1.22}$$

where ϵ^{ijk} is totally antisymmetric and $\epsilon^{123}=1$. These matrices define the **vector representation** of the Lie algebra $so(3)$ and even of the group $SO(3)$. The generators of $SO(3)$ given above satisfy the Lie algebra

$$so(3) := Lie(SO(3)): \quad [T^i, T^j] = i\epsilon^{ijk}T^k, \tag{1.23}$$

which is exactly the same as that for $su(2)$ found above. This follows either by explicitly computing the commutators or by using the epsilon relation (the sum over k is implied)

$$\epsilon^{ijk}\epsilon^{mnk} = \delta^{im}\delta^{jn} - \delta^{in}\delta^{jm}. \tag{1.24}$$

Then

$$\begin{aligned}([T^i,T^j])_{mn} &= (T^i)_{mp}(T^j)_{pn} - (i\leftrightarrow j) = (-i)^2(\epsilon^{imp}\epsilon^{jpn} - (i\leftrightarrow j)) \\ &= -(-i)^2((\delta^{ij}\delta^{mn} - \delta^{in}\delta^{jm}) - (\delta^{ji}\delta^{mn} - \delta^{jn}\delta^{im})) = \delta^{im}\delta^{jn} - \delta^{in}\delta^{jm},\end{aligned} \tag{1.25}$$

which is exactly what we get when evaluating the right hand side of the Lie algebra

$$i\epsilon^{ijk}(T^k)_{mn} = i(-i)\epsilon^{ijk}\epsilon^{kmn} = \delta^{im}\delta^{jn} - \delta^{in}\delta^{jm}. \tag{1.26}$$

The above way of writing the $so(3)$ generators in terms of the epsilon symbol is clearly unique to three dimensions. There is, however, a different way to express these generators that directly generalises to the rotation group $SO(N)$ in any dimension N, namely

$$(T^{ij})_{kl} = -2i\delta^{ij}_{kl} := -i(\delta^i_k\delta^j_l - \delta^i_l\delta^j_k), \tag{1.27}$$

which provides one generator for the rotation in each possible plane given by (ij) in $\mathbf{R}^N$. For $N=3$ one finds that these coincide, i.e., $T^1 = T^{23}$ etc. The $so(N)$ Lie algebra reads then

$$[T^{ij}, T^{kl}] = -i(\delta^{jk}T^{il} + \delta^{il}T^{jk} - \delta^{jl}T^{ik} - \delta^{ik}T^{jl}). \tag{1.28}$$

[8] Note that these matrices are automatically traceless which means that the group $O(N)$ has the same set of generators. The difference between $O(N)$ and $SO(N)$ is thus related to discrete transformations only contrary to the situation for unitary groups where the determinant condition eliminates one generator, namely the $U(1)$ part of the group $U(N)$.

Above we saw that both the 2 by 2 Pauli matrices (multiplied by a factor $\frac{1}{2}$) and the 3 by 3 matrices $(T^i)_{jk} = -i\epsilon^{ijk}$ satisfy the same Lie algebra, namely $su(2) \approx so(3)$. The spin 1/2 representation (the Pauli matrices) are fundamental since any other spin j representation can be constructed from them. An example is provided by the well-known (from QM) formula $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$ which tells us that the tensor product of two spin 1/2 representations gives one spin 1 and one spin 0 representation. The result of the tensor product is therefore called reducible since it can be split into two smaller representations $1 \oplus 0$. Spin j representations, on the other hand, cannot be split into smaller ones and are thus called irreducible often referred to as **irreps** (irreducible representations). In fact, if we consider two-component wave functions χ_α to describe spin 1/2 particles then one can make the above tensor product explicit by writing

$$\tfrac{1}{2} \otimes \tfrac{1}{2} = 1 \oplus 0 \ \ \leftrightarrow \chi_\alpha \chi_\beta = \tfrac{1}{2}\epsilon_{\alpha\beta}(\bar{\chi}\chi) + \tfrac{1}{2}\sigma^i_{\alpha\beta}(\bar{\chi}\sigma^i\chi), \tag{1.29}$$

where $\bar{\chi}\chi := \bar{\chi}^\alpha \chi_\alpha$ with $\bar{\chi}^\alpha := \epsilon^{\alpha\beta}\chi_\beta$ and similarly for $\bar{\chi}\sigma^i\chi = \bar{\chi}^\alpha (\sigma^i)_\alpha{}^\beta \chi_\beta$. Note that the Pauli matrices with both indices down are defined by

$$\sigma^i_{\alpha\beta} := \epsilon_{\beta\gamma}\sigma^i_\alpha{}^\gamma, \tag{1.30}$$

and are easily seen to be symmetric matrices. These together with the antisymmetric matrix $\epsilon_{\alpha\beta}$ thus span the space of 2 by 2 matrices.

In the above tensor product it is important that the expansion coefficients are numerically invariant tensors. That the matrix $\epsilon_{\alpha\beta}$ is numerically invariant under $su(2)$ follows from

$$g \in SU(2) \Rightarrow g_\alpha{}^\gamma g_\beta{}^\delta \epsilon_{\gamma\delta} = (\det g)\, \epsilon_{\alpha\beta} = \epsilon_{\alpha\beta}, \tag{1.31}$$

using first the definition of the determinant and then the fact that any $g \in SU(2)$ has unit determinant [9]. That also the Pauli matrices are numerically invariant is a bit more tricky to show but is done below.

We now show that the Pauli matrices are numerically **invariant** under rotations, not just covariant tensors. By considering their index structure $(\sigma^i)_\alpha{}^\beta$, i.e., one vector index i and two spinor indices, a lower α and a upper β, and then by "rotating" all three we can prove that the effect of all three rotation matrices cancel out completely and we get back just the Pauli matrices. We consider here the rather simple case of rotating the two Pauli matrices $(\sigma^1)_\alpha{}^\beta$ and $(\sigma^2)_\alpha{}^\beta$ around the *z-axis*. A general tensor X with the same index structure transforms as $(X^i)_\alpha{}^\beta \to ((X^i)_\alpha{}^\beta)' := R^i{}_j(\theta)(R_\alpha{}^\gamma(\theta)(X^j)_\gamma{}^\delta(R^{-1}_\delta{}^\beta(\theta))$. Thus if we consider the special case $(X^i)_\alpha{}^\beta = (\sigma^i)_\alpha{}^\beta$ we have

$$(\sigma^i)_\alpha{}^\beta \to ((\sigma^i)_\alpha{}^\beta)' := R^i{}_j(\theta)(R_\alpha{}^\gamma(\theta)(\sigma^j)_\gamma{}^\delta(R^{-1}_\delta{}^\beta(\theta)), \tag{1.32}$$

so the computation we need to do is to evaluate the right hand side. Note that we here also include the third component of the Pauli matrices. For rotations around the *z-axis* the rotation matrices read

$$R(\theta) = e^{i\theta T^3}, \tag{1.33}$$

[9] Compare to how the flat space-time metric behaves under Lorentz transformations.

where T^3 is one of the generators satisfying the $so(3)$ algebra above $[T^i, T^j] = i\epsilon^{ijk}T^k$. Both the spin 1 and spin 1/2 representations are given above and we recall that

$$\textit{spin } 1: \;\; (T^3)^i{}_j = -i\epsilon^{3ij} = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \;\; \textit{spin } 1/2: \;\; (T^3)_\alpha{}^\beta = \frac{1}{2}(\sigma^3)_\alpha{}^\beta. \tag{1.34}$$

The rotation matrices then become in the vector representation

$$\textit{spin } 1: \;\; (R(\theta))^i{}_j = \exp\theta \left(\begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.35}$$

and in the spinor representation

$$\textit{spin } 1/2: \;\; (R(\theta))_\alpha{}^\beta = \exp\left(i\tfrac{\theta}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right) = \cos(\frac{\theta}{2})\delta_\alpha^\beta + i(\sigma^3)_\alpha{}^\beta \sin(\frac{\theta}{2}). \tag{1.36}$$

Applying these transformations to the Pauli matrices gives

$$\textit{spin } 1: \;\; (R(\theta))^i{}_j\, \sigma^j = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma^1 \\ \sigma^2 \\ \sigma^3 \end{pmatrix} = \begin{pmatrix} \cos\theta\sigma^1 + \sin\theta\sigma^2 \\ -\sin\theta\sigma^1 + \cos\theta\sigma^2 \\ \sigma^3 \end{pmatrix}, \tag{1.37}$$

which we now should compare to

$$R_\alpha{}^\gamma(\theta)(\sigma^i)_\gamma{}^\delta(R_\delta^{-1\beta}(\theta) = \left(\exp\left(i\tfrac{\theta}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right)\sigma^i\, \exp\left(-i\tfrac{\theta}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right)\right)_\alpha{}^\beta. \tag{1.38}$$

For σ^3 we thus we see that since σ^3 commutes with itself the result is just σ^3 since the two exponential factors cancel each other. For σ^1 and σ^2, on the other hand, they anticommute with σ^3 so flipping the order of the matrices (expand the *exp* in a power series) gives

$$R_\alpha{}^\gamma(\theta)(\sigma^1)_\gamma{}^\delta(R_\delta^{-1\beta}(\theta) = (\exp i\tfrac{\theta}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\sigma^1\, \exp\left(-i\tfrac{\theta}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right))_\alpha{}^\beta = (\exp i\theta\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\sigma^1)_\alpha{}^\beta, \tag{1.39}$$

which thus gives

$$R(\theta)(\sigma^1)R^{-1}(\theta) = (\cos\theta + i\sigma^3\sin\theta)\sigma^1 = \cos\theta\sigma^1 - \sin\theta\sigma^2, \tag{1.40}$$

while for σ^2 we get

$$R(\theta)(\sigma^2)R^{-1}(\theta) = (\cos\theta + i\sigma^3\sin\theta)\sigma^2 = \cos\theta\sigma^2 + \sin\theta\sigma^1. \tag{1.41}$$

From these results we conclude that

$$R(\theta)(\sigma^i)R^{-1}(\theta) = (R^{-1})^i{}_j\,(\sigma^j), \tag{1.42}$$

and hence that

$$(\sigma^i)_\alpha{}^\beta \to ((\sigma^i)_\alpha{}^\beta)' := R^i{}_j(\theta)(R_\alpha{}^\gamma(\theta)(\sigma^j)_\gamma{}^\delta R_\delta^{-1\beta}(\theta)) = (\sigma^i)_\alpha{}^\beta, \tag{1.43}$$

as expected. This proof will be done in the course in complete generality using the Lie algebra instead which simplifies it a lot (see below).

1.4.2 Lie algebras: the non-compact case

In four-dimensional space-time we need to implement Lorentz symmetry, i.e., $SO(1,3)$. This Lie algebra can be obtained as follows. The $SO(3)$ generators above can be rewritten in a way that is valid in any dimension by setting $T^1 := T^{23}$ etc, that is we index the generators by the plane (for T^1 the (23)-plane) in which they generate a rotation:

$$(T^{ij})_{mn} = -2i\delta^{ij}_{mn} : \quad (T^{12})_{mn} = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad etc. \tag{1.44}$$

This can then be done in any dimension and the definition $(T^{ij})_{mn} = -2i\delta^{ij}_{mn}$ is true for any dimension d. However, when writing the $SO(N)$ Lie algebra this way we find

$$[T^{ij}, T^{kl}] = -i(\delta^{jk}T^{il} + ...) \tag{1.45}$$

where the three terms not written are such that the anti-symmetry in the two pairs of indices ij and kl are implemented. This can be checked in three dimensions by considering one specific example: using e.g. the commutator between $T^1 = T^{23}$ and $T^2 = T^{31}$ must give $iT^3 = iT^{12} = -iT^{21}$ using either form of the Lie algebra.

For space-time signature $(+,-,-,-)$ we use the new T^{ij} form of the $SO(3)$ algebra and let $-\delta^{jk}$ be the space-space part of $\eta_{\mu\nu}$ (denoted $g_{\mu\nu}$ in Peskin and Schroeder) and hence we conclude that

$$Lie(SO(1,3): \quad [J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho}J^{\mu\sigma} + ...), \tag{1.46}$$

where we call the Lorentz generators $J^{\mu\nu}$ (instead of $T^{\mu\nu}$) and explicitly we have shown that the **vector representation**

$$(J^{\mu\nu})_{\rho\sigma} = 2i\delta^{\mu\nu}_{\rho\sigma} := i(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}). \tag{1.47}$$

satisfy the Lorentz algebra given above.

In this representation we can explicitly see that the Minkowski metric is an invariant (i.e., numerically invariant) tensor under $SO(1,3)$. A rotation with finite parameters $\omega_{\mu\nu}$ is defined as acting with the group element[10]

$$g \in SO(1,3): \quad g = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} \tag{1.48}$$

which in the vector representation reads

$$g \in SO(1,3): \quad g_{\rho}{}^{\sigma} = e^{-\frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})_{\rho}{}^{\sigma}} = e^{\omega_{\rho}{}^{\sigma}}. \tag{1.49}$$

Acting on a vector V_μ (the same is true for V^μ) this becomes

$$g_{\mu}{}^{\nu}V_{\nu} = V_{\mu} + \omega_{\mu}{}^{\nu}V_{\nu} + ... \tag{1.50}$$

[10]The factor $\frac{1}{2}$ in the exponent is needed since the sum is over two indices.

Defining the infinitesimal Lorentz transformation (i.e., small ωs) by the variation

$$\delta_\omega V_\mu := (g_\mu{}^\nu V_\nu - V_\mu)|_{\mathcal{O}(\omega)} = \omega_\mu{}^\nu V_\nu, \tag{1.51}$$

we see that $\omega_{01} = -\omega_{10} := \beta$ gives a boost in the negative x direction

$$\delta_\beta V_0 = -\beta V_1, \;\; \delta_\beta V_1 = -\beta V_0, \tag{1.52}$$

while for V^μ it would be in the positive x direction. Note that this way $V^0 V_0 + V^1 V_1$ is invariant as it must.

We can now verify that also the Minkowski metric is invariant although it is a tensor

$$\delta(\eta_{\mu\nu}) := \omega_\mu{}^\rho \eta_{\rho\nu} + \omega_\nu{}^\rho \eta_{\mu\rho} = \omega_{\mu\nu} + \omega_{\nu\mu} = 0. \tag{1.53}$$

Demanding invariance of a tensor like this is, in fact, another way to define the Lie algebra in question. Similar to the rotation invariance of Pauli matrices above the Dirac matrices $(\gamma^\mu)_a{}^b$ are Lorentz invariant.

Comments:
1. We saw above that the Lie algebras of $SU(2)$ and $SO(3)$ are identical, i.e., *isomorphic*. This happens only in a very small number of cases and only for groups with small dimension.
2. On the other hand, that these two Lie algebras are the same shows that the Pauli matrices and the epsilon tensor furnish two *inequivalent* representations, of dimension 2 and 3, respectively, of one and the same Lie algebra. There is an infinite number of finite-dimensional matrix representations of any of the Lie algebras in the Cartan classification.
3. These two representations are both unitary.
4. Another case is $Lie(SO(1,3)) = Lie(SL(2,\mathbf{C}))$ which is related to the fact that Weyl spinors ψ_L and ψ_R are *inequivalent* two-dimensional *complex* representations of the Lie algebra of the Lorentz group. $SL(2,\mathbf{C})$ is the group of all two by two complex matrices with unit determinant. In the Cartan classification its Lie algebra belongs to the class denoted A_2. In the case of $Lie(SO(1,3))$ none of the infinite set of finite-dimensional representations is unitary and are thus irrelevant in quantum mechanics and QFT. The Lorentz group is non-compact and can thus only be implemented unitarily on infinite-dimensional vector spaces, i.e. on Hilbert spaces, a well-known theorem in mathematics.