

1.5 Group theory 3: Yang-Mills theory

As we have seen for both complex scalar and Dirac spinor fields, since they are complex valued they can carry a charge related to an abelian gauge field, a Maxwell field which interact with them through the covariant derivative. In the Lagrangian this gauge field is represented by its own kinetic term $\mathcal{L}(A_\mu) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. In the Standard Model such a Maxwell field appears before the Higgs effect/symmetry breaking and its charge is called *hypercharge*. The EM we study in this course on *QFT* and in other courses arises after the Higgs effect/symmetry breaking as a combination of the hypercharge field and another field.

The most efficient way to obtain the Maxwell field strength is to consider the covariant derivative $D_\mu = \partial_\mu - ieA_\mu$ acting on a complex field. Then the commutator of two such derivatives is also a covariant object. Acting on a complex scalar field Φ we get

$$\begin{aligned} [D_\mu, D_\nu]\Phi &= ((\partial_\mu - ieA_\mu)(\partial_\nu - ieA_\nu) - (\partial_\nu - ieA_\nu)(\partial_\mu - ieA_\mu))\Phi \\ &= ((\partial_\mu\partial_\nu - \partial_\nu\partial_\mu) - ie(\partial_\mu A_\nu - \partial_\nu A_\mu - A_\mu\partial_\nu - A_\nu\partial_\mu) - e^2(A_\mu A_\nu - A_\nu A_\mu))\Phi. \end{aligned} \quad (1.54)$$

In this result we find, however, that the terms with two derivatives cancel and the terms with two gauge fields also cancel since A_μ is here just the ordinary Maxwell potential. The remaining terms with one gauge field and one derivative can be simplified to

$$[D_\mu, D_\nu]\Phi = -ie(\partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu\partial_\nu - A_\nu\partial_\mu)\Phi = -ieF_{\mu\nu}\Phi \quad (1.55)$$

where

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \quad (1.56)$$

is the standard Maxwell field strength. Note that in the middle expression in (1.55) the derivatives in the first two terms act on everything on their right, i.e., including the field Φ while in the last expression the derivatives in $F_{\mu\nu}$ do not act on Φ just on the gauge field A_μ .

We will now repeat this for a gauge theory which involves non-abelian gauge symmetries. Note that exactly this calculation can also be carried out in the case of general relativity and there the result is the Riemann tensor. The Standard Model contains two such gauge theories: QCD using the gauge group $SU(3)$ and the weak interactions using $SU(2)$. In the latter case we can see this by noting that if we make the thought experiment of turning off the electric charge of the proton then its properties (mass, spin etc) become identical to those of the neutron. This means that in constructing a field theory for the (p, n) system it should be invariant if we "rotate" one of these Dirac fields into the other by a 2×2 unitary matrix belonging to the group $SU(2)$. Thus we should put the two Dirac spinors for these two fermions into a two-dimensional complex representation of $SU(2)$:

$$\Psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}. \quad (1.57)$$

In the Standard Model this construction is instead realised on the constituents of the proton and neutron, i.e., the u and d quarks and furthermore only on their left-handed parts (which is possible since the quarks are massless before symmetry breaking). Thus, in the Standard Model we have particles like

$$\Psi_L(\text{quark}) = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad \Psi_L(\text{lepton}) = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \quad (1.58)$$

where we have simplified the notation by writing u_L instead of $\psi_L(u)$ etc. The corresponding right-handed spinors transform as scalars under this $SU(2)$.

The gauge field in this case is then introduced in the usual fashion

$$D_\mu \Psi := \partial_\mu \Psi - \frac{i}{2} g \sigma^i A_\mu^i \Psi = \partial_\mu \Psi - ig T^i A_\mu^i \Psi = \partial_\mu \Psi - ig A_\mu \Psi, \quad (1.59)$$

where g is the coupling constant and $T^i := \frac{1}{2} \sigma^i$ the generators of the Lie algebra $su(2)$

$$[T^i, T^j] = i\epsilon^{ijk} T^k \quad (1.60)$$

expressed as usual in terms of the Pauli matrices satisfying $[\sigma^i, \sigma^j] = 2i\epsilon^{ijk} \sigma^k$. In the covariant derivative each of the three generators is associated with a (real valued) vector potential denoted A_μ^i . In the last form of the covariant derivative we have defined the very useful *matrix valued* vector potential $A_\mu := T^i A_\mu^i$. Note that here the coupling constant g will follow the gauge field A_μ^i while the generators T^i will only depend on the representation of the field acted on by D_μ and the matrix valued A_μ .

As we will now show the above covariant derivative is tied to the gauge transformation

$$\Psi \rightarrow \Psi' = e^{\frac{i}{2} \alpha^i \sigma^i} \Psi := U \Psi, \quad (1.61)$$

which is an $SU(2)$ gauge transformation since $U := e^{\frac{i}{2} \alpha^i \sigma^i} \in SU(2)$ with parameters $\alpha^i(x)$. Recall that $\frac{1}{2} \sigma^i$ satisfy the Lie algebra of $SU(2)$. Demanding gauge invariance (that is covariance under local transformations with $\alpha^i = \alpha^i(x)$) we find that we need the gauge field variation to read

$$\delta A_\mu^i = \frac{1}{g} D_\mu \alpha^i = \frac{1}{g} \partial_\mu \alpha^i + \epsilon^{ijk} A_\mu^j \alpha^k. \quad (1.62)$$

This will then lead to a field strength

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \quad (1.63)$$

or

$$F_{\mu\nu}^i := \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g\epsilon^{ijk} A_\mu^j A_\nu^k. \quad (1.64)$$

The above features of non-abelian gauge theory, i.e., Yang-Mills theory, will now be shown to follow as usual from gauge invariance which here reads

$$\Psi \rightarrow \Psi' = e^{\frac{i}{2} \alpha^i \sigma^i} \Psi = U \Psi \implies D_\mu \Psi \rightarrow (D_\mu \Psi)' = e^{\frac{i}{2} \alpha^i \sigma^i} D_\mu \Psi = U D_\mu \Psi, \quad (1.65)$$

where, using the matrix valued notation introduced above,

$$(D_\mu \Psi)' := \partial_\mu(U\Psi) - igA'_\mu U\Psi. \quad (1.66)$$

Covariance then implies

$$(D_\mu \Psi)' := \partial_\mu(U\Psi) - igA'_\mu U\Psi = UD_\mu \Psi = U(\partial_\mu - igA_\mu)\Psi. \quad (1.67)$$

Since $\partial_\mu(U\Psi) = (\partial_\mu U)U^{-1}U\Psi + U\partial_\mu \Psi$ the above equation implies

$$-igA'_\mu = -igUA_\mu U^{-1} - (\partial_\mu U)U^{-1}. \quad (1.68)$$

This can be written nicely as (use $(\partial_\mu U)U^{-1} = -U\partial_\mu U^{-1}$)

$$A'_\mu = \frac{i}{g}UD_\mu U^{-1} = \frac{i}{g}U(\partial_\mu - igA_\mu)U^{-1}. \quad (1.69)$$

Next we want to read off the field variation of the vector potential from this result. Note that to first order in the parameter α^i the above result for A'_μ gives the variation

$$\begin{aligned} \delta A_\mu &:= (A'_\mu - A_\mu)|_{\mathcal{O}(\alpha)} = \frac{i}{g}(1 + i\alpha^i T^i)D_\mu((1 - i\alpha^i T^i)|_{\mathcal{O}(\alpha)} - A_\mu) \\ &= (\frac{1}{g}\partial_\mu \alpha + A_\mu + i[\alpha, A_\mu])|_{\mathcal{O}(\alpha)} - A_\mu = \frac{1}{g}(\partial_\mu \alpha - ig[A_\mu, \alpha]) = \frac{1}{g}D_\mu \alpha \end{aligned} \quad (1.70)$$

where we in the last expressions have introduced also a matrix valued parameter $\alpha := \alpha^i T^i$. Note also that in the very last form the covariant derivative acts on the parameter α which is Lie algebra valued (as is the case also for A_μ) and therefore leads to a commutator since $\alpha := \alpha^i T^i$ is a matrix and thus involves two fundamental representations corresponding to its two matrix indices! It is important to realize that quite generally a *variation* of a gauge field is a covariant quantity.

Evaluating the variation in terms of the Lie algebra we can use the fact that for two Lie algebra valued quantities (hermitian ones satisfying $[T^i, T^j] = if^{ijk}T^k$) we have

$$[\alpha, \beta] = \alpha^i \beta^j [T^i, T^j] = \alpha^i \beta^j if^{ijk}T^k. \quad (1.71)$$

This then implies (using that the structure constants are antisymmetric in all three indices)

$$\delta A_\mu^i = \frac{1}{g}(D_\mu \alpha)^i = \frac{1}{g}(\partial_\mu \alpha^i + gf^{ijk}A_\mu^j \alpha^k). \quad (1.72)$$

This is the result that was quoted above for $SU(2)$ for which $f^{ijk} = \epsilon^{ijk}$.

The field strength can now be computed in the same way as in the abelian case described in the beginning of this subsection, now keeping the terms quadratic in the matrix valued gauge fields. This gives directly

$$[D_\mu, D_\nu]\Psi = -ig(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu])\Psi, \quad (1.73)$$

where special care has been taken to the matrix properties of the field A_μ . Defining the field strength as the expression in the bracket above multiplying Ψ we get

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \quad (1.74)$$

or

$$F_{\mu\nu}^i := \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + gf^{ijk}A_\mu^j A_\nu^k. \quad (1.75)$$