1.10.5 The quantisation of the Dirac field and the Dirac propagator

The quantisation of the Dirac field is on the surface very similar to the case of the scalar fields discussed previously. There are, however, some new subtle points here. Let us recall the mode expansion found before

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \Sigma_s \left(a^s_{\mathbf{p}} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}} v^s(p) e^{ip\cdot x} \right), \tag{1.380}$$

One new feature is that this field must be **anti-commuting**. For the field ψ this property can be taken care of by defining the operators to anti-commute, that is, if their order is flipped one gets a minus sign $a^s_{\mathbf{p}} a^{s'}_{\mathbf{p}'} = -a^{s'}_{\mathbf{p}'} a^s_{\mathbf{p}}$, or $\{a^s_{\mathbf{p}}, a^{s'}_{\mathbf{p}'}\} = 0$, and similarly for all other choices of spin 1/2 operators. Of course, if they give a delta function the relation is instead

$$\{a^s_{\mathbf{p}}, a^{s'\dagger}_{\mathbf{p}'}\} = (2\pi)^3 \delta^3(\mathbf{p}' - \mathbf{p}) \delta^{ss'}. \tag{1.381}$$

The notation for the operators is very similar to the one for the scalar fields but there is no relation whatsoever between them! (The s will help you keep them apart.)

There are several reasons for this anti-commutivity. Even before the quantisation of ψ we saw that for Majorana spinors the mass term is $\bar{\psi}\psi = \psi_a C^{ab} \psi_b$ which is only non-vanishes if the spinors anti-commute (recall that the matrix C is antisymmetric), i.e., $\psi_a \psi_b = -\psi_b \psi_a$. This conclusion also follows from the kinetic term for Majorana spinors. Are there both bosonic (commuting) fields and fermionic (anti-commuting) ones involved in the reordering of fields then minus signs are generated only when a fermion is moved from one side to the other of another fermion.

Exercise: Prove the last statement about the kinetic term for Majorana spinors. This involves first showing that the matrices $C\gamma^\mu$ are symmetric[32].

Comment: Anti-commuting "coordinates" is mathematically not a problem and is a well-established aspect of superspace (in supergravity), supergroups etc, research fields developed first by physicists. The unusual aspect of classical objects satisfying $ab = -ba$ is that they cannot be represented by numbers and thus have no size, only direction. E.g., distance has no meaning in the super-directions in superspace.

Three other physical arguments for the anti-commuting property of ψ are

1. The Pauli exclusion principle says that no state can have two exactly the same spin 1/2 particle in it. This implies that the two-particle state $a^{s\dagger}_{\mathbf{p}} a^{s\dagger}_{\mathbf{p}} |0\rangle = 0$. This is probably familiar to you from QM. To implement this property one just declares that the operators anti-commute $a^s_{\mathbf{p}} a^{s'}_{\mathbf{p}'} = -a^{s'}_{\mathbf{p}'} a^s_{\mathbf{p}}$ since then by setting $s = s'$ and the momenta equal this equation says $aa = -aa \Rightarrow aa = 0$.

2. Non-negative energy, $E \geq 0$, implies that the operators must anti-commute as will be demonstrated later.

3. Causality also implies an anti-cummuting spinor field ψ as we know show. Consider

[32] This is a crucial fact in supersymmetry.

for this purpose the retarded Dirac propagator which we just write down by analogy with the scalar case:

$$S_R(x_2 - x_1) := \Theta(t_2 - t_1)\langle 0|\{\psi(x_2), \bar{\psi}(x_1)\}|0\rangle. \tag{1.382}$$

Note: The Dirac conjugate field is the one to the right since this propagator is a matrix in spinor space. And we have here assumed that ψ anti-commutes which is the reason for the anti-commutator.

We will now compute this retarded propagator to see in detail why it is only causal if defined with an anti-commutator. The first term involves only a and $a^\dagger$ since (recall that ψ contains a and $b^\dagger$ while $\bar{\psi}$ contains b and $a^\dagger$)

$$\langle 0|\psi(x_2)\bar{\psi}(x_1)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\frac{1}{\sqrt{2E'_{\mathbf{p}'}}}\Sigma_{s,s'}\langle 0|a^s_{\mathbf{p}}e^{-ip\cdot x_2}u^s(p)\bar{u}^{s'}(p')a^{s'\dagger}_{\mathbf{p}'}e^{ip'\cdot x_1}|0\rangle. \tag{1.383}$$

The anti commutation relations $\{a^s_{\mathbf{p}}, a^{s'\dagger}_{\mathbf{p}'}\} = (2\pi)^3\delta^3(\mathbf{p}'-\mathbf{p})\delta^{ss'}$ then means that we can use the delta functions to do the $\mathbf{p}'$ integrals. Note, however, that we can wait with declaring if the delta functions come from anti commutators or commutators. In any case this gives

$$\langle 0|\psi(x_2)\bar{\psi}(x_1)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x_2-x_1)}\Sigma_s u^s(p)\bar{u}^s(p) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x_2-x_1)}(\gamma\cdot p+m). \tag{1.384}$$

It is now possible to bring the Dirac operator outside the integral if we first replace the momentum in it with a derivative. Choosing a derivative on x_2 this step leads to

$$\langle 0|\psi(x_2)\bar{\psi}(x_1)|0\rangle = (i\gamma\cdot\partial_{(x_2)} + m)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x_2-x_1)}. \tag{1.385}$$

Repeating this calculation for the second term in the retarded propagator we get a similar result, apart from an overall minus sign,

$$\langle 0|\bar{\psi}(x_1)\psi(x_2)|0\rangle = -(i\gamma\cdot\partial_{(x_2)} + m)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x_1-x_2)}. \tag{1.386}$$

Note that the derivative is still on the x_2 coordinates but that the order of x_1 and x_2 in the exponent has changed.

To get causality when we combine these two results we should aim at getting the Dirac operator to act on the scalar field retarded propagator. This can only happen if the two results are *added* since then the minus sign in the second result gives rise to the retarded scalar field propagator which does contain a relative minus sign. Thus we have shown causality for anti-commuting spin 1/2 field and that the result can be written (note the plus sign)

$$S_R(x_2 - x_1) = (i\gamma\cdot\partial_{(x_2)} + m)D_R(x_2 - x_1). \tag{1.387}$$

The equation that defines the spin 1/2 Green's function is obtained from the Dirac equation (note the minus sign)

$$(i\gamma\cdot\partial_{(x_2)} - m)S_R(x_2 - x_1) = i\delta^4(x_2 - x_1). \tag{1.388}$$

This is verified by inserting the above equation for $S_R(x_2 - x_1)$ which gives

$$(i\gamma\cdot\partial_{(x_2)}-m)(i\gamma\cdot\partial_{(x_2)}+m)D_R(x_2-x_1) = -(\Box_{(x_2)}+m^2)D_R(x_2-x_1) = i\delta^4(x_2-x_1). \quad (1.389)$$

A general Green's function is therefore determined by its Fourier transform (use $e^{-ip\cdot x}$)

$$\tilde{S}(p) = \frac{i}{\gamma\cdot p - m} := \frac{i(\gamma\cdot p + m)}{p^2 - m^2}, \quad (1.390)$$

where the expression in the middle is common but only formal (matrix in the denominator) and is defined by the last expression.

As in the scalar case we get the Feynman propagator by adding $i\epsilon$ in the denominator

$$\tilde{S}(p) = \frac{i}{\gamma\cdot p - m} := \frac{i(\gamma\cdot p + m)}{p^2 - m^2} \Rightarrow \tilde{S}_F(p) = \frac{i(\gamma\cdot p + m)}{p^2 - m^2 + i\epsilon}, \quad (1.391)$$

which gives

$$S_F(x_2-x_1) = \langle 0|T(\psi(x_2)\bar{\psi}(x_1))|0\rangle := \Theta(t_2-t_1)\langle 0|\psi(x_2)\bar{\psi}(x_1)|0\rangle - \Theta(t_1-t_2)\langle 0|\bar{\psi}(x_1)\psi(x_2)|0\rangle. \quad (1.392)$$

It is important to note that the minus sign between the two terms is a consequence of the facts that the time ordering implies changing the order of the operators when $t_2 < t_1$ and that the ψ fields anti-commute.

With the quantum Dirac field at hand we can now compute various operators of interest like the Hamiltonian, charge and angular momentum operators.

Starting with the Hamiltonian we must first derive it from the Lagrangian. The standard way to do this is by a Legendre transformation. However, for Lagrangians which are first order in time derivatives this step is not necessary since $\mathcal{L}$ is already in this form:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi = \psi^\dagger(i\partial_t + i\gamma^0\gamma^i\partial_i - m\gamma^0)\psi := i\psi^\dagger\dot{\psi} - \mathcal{H}. \quad (1.393)$$

Thus the Hamiltonian is

$$\mathcal{H} = \psi^\dagger(-i\gamma^0\gamma^i\partial_i + m\gamma^0)\psi = \bar{\psi}(-i\gamma^i\partial_i + m)\psi. \quad (1.394)$$

Since the operator in $\bar{\psi}$ are $a^\dagger$ and b they will end up to the left of the operators a and $b^\dagger$ coming from ψ. So using the mode expansions of the quantum Dirac field and performing the same steps as for the scalar fields will give the result

$$\mathcal{H} = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}}\Sigma_s(a^{s\dagger}_{\mathbf{p}}a^s_{\mathbf{p}} - b^s_{\mathbf{p}}b^{s\dagger}_{\mathbf{p}}). \quad (1.395)$$

Here we discover another reason for the anticommuting feature of the Dirac field. The second term must give a positive contribution to the energy after normal ordering which implies that b and $b^\dagger$ must anticommute and give an extra minus sign when their order is flipped. The normal ordered Hamiltonian is therefore

$$:\mathcal{H} := \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}}\Sigma_s(a^{s\dagger}_{\mathbf{p}}a^s_{\mathbf{p}} + b^{s\dagger}_{\mathbf{p}}b^s_{\mathbf{p}}), \quad (1.396)$$

which is zero on the vacuum and positive definite on any other state.

Next we consider the charge operator Q which is obtained from the conserved current for a Dirac field that was derived above:

$$j^\mu = e\bar{\psi}\gamma^\mu\psi \Rightarrow Q = \int d^3r j^0 = e\int d^3r\bar{\psi}\gamma^0\psi = e\int d^3r\psi^\dagger\psi. \tag{1.397}$$

Inserting the mode expansions for $\psi^\dagger(x)$ and $\psi(x)$, doing the space integrals, followed by one of the momentum integrals and using the $v^\dagger u$ type scalar products obtained earlier give directly $Q = e\int\frac{d^3p}{(2\pi)^3}\Sigma_s(a_{\mathbf{p}}^{s\dagger}a_{\mathbf{p}}^s + b_{\mathbf{p}}^s b_{\mathbf{p}}^{s\dagger})$ which after normal ordering becomes

$$: Q := e\int\frac{d^3p}{(2\pi)^3}\Sigma_s(a_{\mathbf{p}}^{s\dagger}a_{\mathbf{p}}^s - b_{\mathbf{p}}^{s\dagger}b_{\mathbf{p}}^s). \tag{1.398}$$

This shows that particles and antiparticles have opposite charge (just let Q act on such states).

Finally we turn to angular momentum which will have an extra subtle point associated with it. From the definition of the Noether current we find that the angular momentum current is

$$j^\mu_{\nu\rho} = \frac{i}{4}\bar{\psi}\gamma^\mu\gamma_{\nu\rho}\psi, \tag{1.399}$$

which gives the charge $M_{\nu\rho}$, and its j_z component, (use that $j_z := M_{12}$)

$$M_{\nu\rho} = \frac{i}{4}\int d^3r\bar{\psi}\gamma^0\gamma_{\nu\rho}\psi \Rightarrow j_z = \int d^3r\,\psi^\dagger(\frac{1}{2}\Sigma_z)\psi, \text{ where } \Sigma_z = \begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}. \tag{1.400}$$

The rest of the calculation follows the pattern form Q above but with Σ_z inserted between the operators. Since also the mixed scalar products $v^\dagger\Sigma_z u$ etc vanish the same way as for Q we get

$$j_z = \int\frac{d^3p'}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}'}}\Sigma_{r,r'}\left(a_{\mathbf{p}'}^{r\dagger}a_{\mathbf{p}'}^{r'}u^{r\dagger}(p')(\frac{1}{2}\Sigma_z)u^{r'}(p') + b_{\mathbf{p}'}^r b_{\mathbf{p}'}^{r'\dagger}v^{r\dagger}(p')(\frac{1}{2}\Sigma_z)v^{r'}(p')\right). \tag{1.401}$$

Note that this vanishes on the vacuum since the first term is normal ordered and the second term gives the trace of the matrix Σ_z which is zero. Thus we can flip the order of $bb^\dagger$ without imposing normal ordering and just pick up a minus sign. To check the implications of this expression we let it act on the one-particle states $a_{\mathbf{p}}^{s\dagger}|0\rangle$ and $b_{\mathbf{p}}^{s\dagger}|0\rangle$ representing a particle and an antiparticle, respectively.

We start with $a_{\mathbf{p}}^{s\dagger}|0\rangle$. Acting with j_z on this state we need $a_{\mathbf{p}'}^{r\dagger}a_{\mathbf{p}'}^{r'}a_{\mathbf{p}}^{s\dagger}|0\rangle = a_{\mathbf{p}'}^{r\dagger}\{a_{\mathbf{p}'}^{r'}, a_{\mathbf{p}}^{s\dagger}\}|0\rangle = (2\pi)^3\delta^3(\mathbf{p}' - \mathbf{p})\delta^{sr'}a_{\mathbf{p}}^{r\dagger}|0\rangle$. Using this result to do the $\mathbf{p}'$ integral and the r' sum in j_z gives

$$j_z a_{\mathbf{p}}^{s\dagger}|0\rangle = \frac{1}{2E_{\mathbf{p}}}\Sigma_r u^{r\dagger}(p)(\frac{1}{2}\Sigma_z)u^s(p)a_{\mathbf{p}}^{r\dagger}|0\rangle. \tag{1.402}$$

To get a feeling for this expression let's evaluate it in the rest frame $p^\mu = (m, 0, 0, 0)$. Then, with $\mathbf{p} = 0$, it reads

$$j_z a_0^{s\dagger}|0\rangle = \frac{m}{2m}\Sigma_r(\xi^{rT}, \xi^{rT})\frac{1}{2}\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}\begin{pmatrix}\xi^s\\ \xi^s\end{pmatrix}a_0^{r\dagger}|0\rangle = \frac{1}{2}\Sigma_r(\xi^{rT}\sigma_z\xi^s)a_0^{r\dagger}|0\rangle. \tag{1.403}$$

Hence we find the expected result for particles:

$$j_z a_0^{s\dagger}|0\rangle = \pm\frac{1}{2}a_0^{s\dagger}|0\rangle, \text{ with } +\frac{1}{2} \text{ for s=1 and } -\frac{1}{2} \text{ for s=2.} \tag{1.404}$$

For antiparticles we evaluate j_z on $b_{\mathbf{p}}^{s\dagger}|0\rangle$ instead. After flipping the order of b and $b^\dagger$ in j_z the so obtained minus sign means

$$j_z b_0^{s\dagger}|0\rangle = \mp\frac{1}{2}b_0^{s\dagger}|0\rangle, \text{ with } -\frac{1}{2} \text{ for s=1 and } +\frac{1}{2} \text{ for s=2.} \tag{1.405}$$

Thus the meaning of the spin direction index s on states is opposite for antiparticles compared to that for particles. Since we here are making this comparison using the same basis ξ^s this is as we should have expected in view of similar things happening for both momentum and charge before. This way the two terms in $\psi(x)$ have the same effect on a general state, i.e., the $a_{\mathbf{p}}^s$-term destroys momentum $\mathbf{p}$ and spin s while the $b_{\mathbf{p}}^{s\dagger}$-term creates momentum $-\mathbf{p}$ and spin $-s$.

1.10.6 Discrete symmetries and the CPT theorem

There are three discrete symmetries that play a special role in QFT:

Parity transformations, or space inversions, $\mathbf{P} : (t, \mathbf{r}) \to (t, -\mathbf{r})$

Time reversal, $\mathbf{T} : (t, \mathbf{r}) \to (-t, \mathbf{r})$

Charge conjugation, $\mathbf{C} : q \to -q$

The theorem we want to prove says that Nature is invariant under the combined transformation CPT but not necessarily under any transformation not involving all three. We know from experiments that Nature is neither invariant under P nor CP, and most likely also not under T if checked independently from CP.

The way to prove the CPT theorem is to write down the most general hermitian and Lorentz invariant Lagrangian and see if it can fail to respect CPT invariance. The discussion here will not be absolutely rigorous but good enough to provide a convincing argument that can be made completely correct.

Experimentally the situation is as follows:
Gravity, EM, QCD: P, C and T are all OK.

Weak interactions without K particles: P and C are not OK, CP and T are OK.

Weak processes involving K: non of C, P and T is OK but CPT seems to be OK.

Comment: Last fact observed by Cronin-Fitch 1964, Nobel prize 1980, requires three families to be possible in QFT. The so called CP violation is needed to understand the matter-antimatter imbalance of the universe.

Comment: Note that for massless spin 1/2 fields $P \Rightarrow \psi_L \to \psi_R$ since the helicity h behaves as $\hat{\mathbf{p}} \cdot (\mathbf{r} \times \mathbf{p})$ so $h \to -h$.

Now we discuss each of the three transformations in turn.

Parity: P
We start by introducing a unitary operator P such that the Dirac operators behave as

$$P a^s_{\mathbf{p}} P^{-1} = \eta_a a^s_{-\mathbf{p}}, \;\; P b^s_{\mathbf{p}} P^{-1} = \eta_b b^s_{-\mathbf{p}}, \tag{1.406}$$

for some complex parameters η_a and η_b. Then the anti-commutation relations imply

$$\eta_a^\star \eta_a = 1, \;\; \eta_b^\star \eta_b = 1, \tag{1.407}$$

so these parameters are just phases. Next we use these parity transformations to see how the Dirac quantum field ψ behaves: we expect the answer to be $\psi(t,\mathbf{r}) \to \psi(t,-\mathbf{r})$. The procedure to show this is similar to how we checked Lorentz invariance for scalar fields earlier, but quite a bit more interesting as we will see:

$$P\psi(t,\mathbf{r})P^{-1} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\Sigma_s(\eta_a a^s_{-\mathbf{p}} u^s(p)e^{-ip\cdot x} + \eta_b^\star b^{s\dagger}_{-\mathbf{p}} v^s(p)e^{ip\cdot x}). \tag{1.408}$$

The next step is to change the integration variable from $\mathbf{p}$ to $\bar{\mathbf{p}} = -\mathbf{p}$. This gives

$$P\psi(t,\mathbf{r})P^{-1} = \int \frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\bar{\mathbf{p}}}}}\Sigma_s(\eta_a a^s_{\bar{\mathbf{p}}}\gamma^0 u^s(\bar{p})e^{-i\bar{p}\cdot\bar{x}} - \eta_b^\star b^{s\dagger}_{\bar{\mathbf{p}}}\gamma^0 v^s(\bar{p})e^{i\bar{p}\cdot\bar{x}}). \tag{1.409}$$

To get to this expression for the parity transformation we have made use of two identities:
1. In the exponents: $p\cdot x = \bar{p}\cdot(t,-\mathbf{r}) = \bar{p}\cdot\bar{x}$.
2. The extra γ^0 matrices arise for a similar reason, namely when replacing p^μ by $\bar{p}^\mu$ we get

$$u(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\xi\\ \sqrt{p\cdot\bar{\sigma}}\xi\end{pmatrix} = \begin{pmatrix}\sqrt{\bar{p}\cdot\bar{\sigma}}\xi\\ \sqrt{\bar{p}\cdot\sigma}\xi\end{pmatrix} = \gamma^0 u(\bar{p}), \text{ while } v(p) = -\gamma^0 v(\bar{p}). \tag{1.410}$$

By choosing $\eta_a = -\eta_b = 1$ we have thus obtained the result that P does give rise to a parity transformation but it is accompanied by a γ^0 matrix:

$$P\psi(t,\mathbf{r})P^{-1} = \gamma^0\psi(t,-\mathbf{r}) \tag{1.411}$$

To understand what this means let's consider a possible factor in a Lagrangian, and its behaviour under parity transformations

$$\bar{\psi}(x)\gamma^{\mu\nu}\psi(x) \to P\bar{\psi}\gamma^{\mu\nu}\psi P^{-1} = P\bar{\psi}P^{-1}\gamma^{\mu\nu}P\psi P^{-1} = \bar{\psi}(\bar{x})\gamma^0\gamma^{\mu\nu}\gamma^0\psi(\bar{x}) = \begin{cases}\bar{\psi}\gamma^{ij}\psi\\ -\bar{\psi}\gamma^{0i}\psi\end{cases} \tag{1.412}$$

By introducing the notation $(-1)^\mu$ which is $= +1$ for $\mu = 0$ and $= -1$ for $\mu = i$ this result can be written

$$P\bar{\psi}(x)\gamma^{\mu\nu}\psi(x)P^{-1} = (-1)^\mu(-1)^\nu\bar{\psi}(\bar{x})\gamma^{\mu\nu}\psi(\bar{x}). \tag{1.413}$$

Time-reversal: T

The operator T has the odd feature of being anti-unitary[33], i.e., it affects the imaginary unit i as follows: $Ti = -iT$, but it is still true that $T^\dagger = T^{-1}$. The action of T on $a^s_{\mathbf{p}}$ is defined to give $a^{-s}_{-\mathbf{p}}$ since the momentum is $m\dot{\mathbf{r}}$ and angular momentum is $\mathbf{r}\times\mathbf{p}$. Then going through the same steps as for parity we get, paying due attention to the fact that T acts also on i,

$$T\psi(t,\mathbf{r})T^{-1} = \gamma^1\gamma^3\psi(-t,\mathbf{r}). \tag{1.414}$$

[33]Recall from ordinary QM that if H is time-reversal invariant, i.e., $[H,T] = 0$, then a time-reversal $t \to -t$ must be compensated for by $i \to -i$ so that the Schroedinger equation is invariant.

An example of how T works is the following (insert $T^{-1}T = 1$)

$$T(\bar{\psi}\gamma^\mu\psi)T^{-1} = (T\bar{\psi}T^{-1})(T\gamma^\mu T^{-1})(T\psi T^{-1}). \tag{1.415}$$

Thus we need two things: 1) From the result $T\psi(t,\mathbf{r})T^{-1} = \gamma^1\gamma^3\psi(-t,\mathbf{r})$ we get

$$T\psi^\dagger(t,\mathbf{r})T^{-1} = \psi^\dagger(-t,\mathbf{r})(\gamma^1\gamma^3)^\dagger = \psi^\dagger(-t,\mathbf{r})(\gamma^3)^\dagger(\gamma^1)^\dagger, \tag{1.416}$$

giving the Dirac conjugate (using the fact that γ^0 is a real matrix)

$$T\bar{\psi}(t,\mathbf{r})T^{-1} = \bar{\psi}(-t,\mathbf{r})\gamma^0(\gamma^3)^\dagger(\gamma^1)^\dagger\gamma^0 = \bar{\psi}(-t,\mathbf{r})\gamma^3\gamma^1, \tag{1.417}$$

and 2) that $T\gamma^\mu T^{-1} = (\gamma^\mu)^\star$ for which we can use the B matrix identity

$$B\gamma^\mu B^{-1} = -(\gamma^\mu)^\star, \text{ with } B = \gamma^2 = -B^{-1} \tag{1.418}$$

which is true since γ^2 is the only complex γ^μ matrix. Using these two results we get

$$T(\bar{\psi}\gamma^\mu\psi)T^{-1} = \bar{\psi}\gamma^3\gamma^1\gamma^2\gamma^\mu\gamma^2\gamma^1\gamma^3\psi = -\bar{\psi}\gamma^1\gamma^2\gamma^3\gamma^\mu\gamma^1\gamma^2\gamma^3\psi, \tag{1.419}$$

which becomes, using $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, i.e., $\gamma^1\gamma^2\gamma^3 = -i\gamma^0\gamma^5$,

$$T(\bar{\psi}\gamma^\mu\psi)T^{-1} = -(-i)^2\bar{\psi}\gamma^0\gamma^5\gamma^\mu\gamma^0\gamma^5\psi = \bar{\psi}\gamma^0\gamma^\mu\gamma^0\psi = (-1)^\mu\bar{\psi}\gamma^\mu\psi. \tag{1.420}$$

Charge conjugation: C

Charge conjugation refers to the operation that flips the charge of the particle, or interchanges the particle with its anti-particle. Thus the action of C is simply

$$Ca^s_{\mathbf{p}}C^{-1} = b^s_{\mathbf{p}}. \tag{1.421}$$

Again using similar steps to above we are led to

$$C\psi(x)C^{-1} = -i\gamma^2\psi^\star(x), \tag{1.422}$$

where it should be clear form the context that C is an operator and not the matrix we have discussed before which was also denoted C. However, taking the last formula one step further we can express it in terms of $\bar{\psi}$ as follows

$$C\psi(x)C^{-1} = -i\gamma^2\psi^\star(x) = -i\gamma^2(\psi^\dagger)^T = -i(\bar{\psi}\gamma^0\gamma^2)^T = (\bar{\psi}C^{-1})^T, \tag{1.423}$$

where C^{-1} on the RHS now is the inverse of the charge conjugation matrix $C = i\gamma^0\gamma^2$ discussed in the context of Majorana spinors previously. Recall $\psi^T C = \bar{\psi} \Rightarrow \psi^T = \bar{\psi}C^{-1}$. Note that for a Majorana spinor this becomes $C\psi(x)C^{-1} = \psi(x)$.

An example here is

$$\begin{aligned}C(\bar\psi\psi)C^{-1} &= (C\psi^\dagger C^{-1})\gamma^0(C\psi C^{-1}) = (-i\gamma^2(\psi^\dagger)^T)^\dagger\gamma^0(-i\gamma^2(\psi^\dagger)^T)\\ &= (\gamma^2(\psi^\dagger)^T)^\dagger\gamma^0\gamma^2(\psi^\dagger)^T = \psi^T(\gamma^2)^\dagger\gamma^0\gamma^2(\psi^\dagger)^T = -\psi^T\gamma^0(\psi^\dagger)^T,\end{aligned} \tag{1.424}$$

where in the last step we got three minus signs from 1) $(\gamma^2)^\dagger = -\gamma^2$, 2) $\gamma^2\gamma^0 = -\gamma^0\gamma^2$ and 3) $(\gamma^2)^2 = -1$. Finally, to get the answer in the standard form (with $\bar\psi$ to the left) we need to flip the order of the two ψs (gives a new minus sign) and use that γ^0 is symmetric. Thus

$$C(\bar\psi\psi)C^{-1} = \bar\psi\psi. \tag{1.425}$$

The computations done above can now be repeated for all possible expressions involving ψ fields and gamma-matrices and used to create the following table, where we have added vectors as partial derivatives ∂_μ and A_μ, together with $F_{\mu\nu}$, and the imaginary unit i. From these building blocks we can construct any possible term in the Lagrangian:

$$\begin{array}{ccccccccc}
\mathcal{O} & \bar\psi\psi & i\bar\psi\gamma^5\psi & \bar\psi\gamma^\mu\psi & \bar\psi\gamma^\mu\gamma^5\psi & \bar\psi\gamma^{\mu\nu}\psi & \partial_\mu,\, A_\mu & F_{\mu\nu} & i\\
P & +1 & -1 & (-1)^\mu & -(-1)^\mu & (-1)^\mu(-1)^\nu & (-1)^\mu & +1 & +1\\
T & +1 & -1 & (-1)^\mu & (-1)^\mu & -(-1)^\mu(-1)^\nu & -(-1)^\mu & +1 & -1\\
C & +1 & +1 & -1 & +1 & -1 & +1 & +1 & +1\\
CPT & +1 & +1 & -1 & -1 & +1 & -1 & +1 & -1
\end{array} \tag{1.426}$$

From this table we can draw the following conclusion:

The CPT theorem: Any term in a hermitian and Lorentz invariant Lagrangian constructed from the objects listed above will be CPT invariant, i.e.,

$$CPT\mathcal{L}(CPT)^{-1} = \mathcal{L}. \tag{1.427}$$

That such a Lagrangian cannot violate CPT invariance can be proven completely rigorously.

The Dirac Lagrangian $\mathcal{L} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi$ is actually invariant under C, P, and T separately which does not have to be true for other terms constructed by multiplying the objects in the table above.

Exercise: Show that the Dirac term $\mathcal{L} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi$ is separately invariant under C, P, and T.