1.11 Interacting field theories and the Feynman diagram expansion

1.11.1 Perturbation theory: an introduction and summary (Chap. 4 in PS)

This is a good point to stop and reflect on what we have done so far and connect it to what will be done in this chapter, namely the development of perturbation theory and Feynman diagram techniques. The final result will be the Feynman rules and formulas needed for the computation of scattering amplitudes, S-matrix elements and cross sections at tree level (Chap 5 in PS) and loop level (the last 1/3 of the course). As we will see the loop diagrams will force us to develop renormalisation techniques.

This chapter (Chapter 4 in PS) contains a lot of information and provides a number of key formulas which we will list below in this introductory discussion. It contains several deep conceptual points which one has to get some understanding of even though they cannot be presented in a completely rigorous way mathematically. One simply has to rely on physics intuition to some extent. The amazing fact is that QFT works extremely well despite this. This is amply shown by the $g-2$ QFT calculation which is in accord with experiments to about 10 decimal points (see David Gross at Solvay 2011[34]).

So far we have discussed classical interacting field theories and applied them to physics, in particular the Higgs mechanism. To understand where we are in the development of quantum field theory and the computation of scattering amplitudes let us first consider a couple of typical and very basic examples of interacting field theories. The fields we will use to construct theories are real and complex scalar fields, $\phi(x)$ and $\Phi(x)$, spin 1/2 fermions, Dirac, Weyl and Majorana, that is $\psi(x)$ without or with some restriction, and the EM spin 1 vector potential A_μ.

One of the simplest theories is ϕ^4, the so called **"fi-to-the-fourth theory"** which will also serve as a playground when discussing more involved issues like renormalisability later. Its Lagrangian reads

$$\mathcal{L}(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{1.428}$$

Later we will also discuss terms cubic in ϕ. A complex version of this theory was studied previously in the Higgs model. Coupling a scalar field to a fermion gives the so called **Yukawa theory**, here without the ϕ^4 term:

$$\mathcal{L}(\phi,\psi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m_\psi) - g\phi\bar{\psi}\psi, \tag{1.429}$$

while coupling a complex scalar to a vector potential is known as **scalar QED**:

$$\mathcal{L}(A_\mu,\Phi) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\Phi^\star D^\mu\Phi - m^2\Phi^\star\Phi. \tag{1.430}$$

Ordinary **QED** is the theory obtained from coupling EM and Dirac fermions:

$$\mathcal{L}(A_\mu,\psi) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi. \tag{1.431}$$

[34] See also the recent research paper quoted in a footnote in the previous chapter on the Dirac theory.

Of course, it is possible to have all these fields appearing in the same Lagrangian which is what is required in the standard model of elementary particles.

Comment: (not part of the course) From the EM vector potential and the field strength one can construct two other terms in a Lagrangian:
1) In four space-time dimensions the action, here in Yang-Mills theory,

$$S = \frac{1}{8\pi^2} \int d^4x \, \epsilon^{\mu\nu\rho\sigma} Tr(F_{\mu\nu} F_{\rho\sigma}). \tag{1.432}$$

This action has exactly the same form even if coupled to gravity and used in a curved manifold. It is a topological term without any degrees of freedom which is a consequence of the fact that it is invariant under $\delta A_\mu = \Lambda_\mu$ so the whole vector potential can be gauged away. A related fact is that $\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$ is a total derivative and hence it does not contribute to the equation of motions in the bulk (recall the discussion on field variations in the beginning of the course).
2) In three space-time dimensions there is another topological expressions, the so called **Chern-Simons theory**,

$$S = \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho}, \tag{1.433}$$

where k is an integer for topological reasons. This theory is also known to lack any degrees of freedom in the bulk (although it is not a total derivative) so all interesting physics takes place on the boundary. Features like this make this theory very important in condensed matter physics, e.g., as a model for the FQHE (fractional quantum Hall effect)[35].

All the field theories we will study have one **free part**, $\mathcal{L}_0$, consisting of all bilinear terms (kinetic plus mass term) and one **interacting part**, $\mathcal{L}_{int}$, built from all terms with three or more fields, i.e.,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}. \tag{1.434}$$

In the Hamiltonian the interaction terms appear with the opposite sign compared to $\mathcal{L}$ so in the above cases we have

$$\mathcal{H}_{int} = \frac{\lambda}{4!}\phi^4, \;\; \mathcal{H}_{int} = g\phi\bar{\psi}\psi \;\; \text{and} \;\; \mathcal{H}_{int} = eA_\mu \bar{\psi}\gamma^\mu\psi, \tag{1.435}$$

while the scalar QED theory was analysed in the context of the Higgs mechanism.

Renormalisibility: The fact that the theories above cannot contain interaction terms with an arbitrary number of fields comes from the requirement of predictibility. When using a Lagrangian to predict the outcome of an experiment the parameters (masses and coupling constants, and in fact also the fields) must have been determined already by other experiments. It is therefore crucial that the Lagrangian has only a finite number of parameters and fields otherwise one would have to do an infinite number of experiments before the theory can be used to predict anything. This philosophy can be turned into a precise statement in terms of **renormalisibility**.

[35] For a review see G.V. Dunne, "Aspects of Chern-Simons theory", hep-th/9902115.

Fortunately, there is a basically trivial criterion (to be proved later) that ensures that a theory has predictive powers: it has to be **renormalisable**. This criterion is expressed in terms of dimensions.

The action functional $S[\phi]$ has the same dimension as $\hbar$. In natural units the action is therefore dimensionless $[S] = L^0$ which implies that the dimension of the Lagrangian density in four space-time dimensions is $[\mathcal{L}] = L^{-4}$. The kinetic terms will then determine the dimension of the fields:

$$[\mathcal{L}] = L^{-4} \Rightarrow [\phi] = L^{-1},\ \ [A_\mu] = L^{-1},\ \ [\psi] = L^{-3/2}. \tag{1.436}$$

Consider now a generic scalar field theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \lambda_3\phi^3 - \lambda_4\phi^4 - \lambda_5\phi^5 - .... \tag{1.437}$$

The parameters in this Lagrangian have the following dimensions:

$$[m^2] = L^{-2},\ \ [\lambda_3] = L^{-1},\ \ [\lambda_4] = L^0,\ \ [\lambda_5] = L^1,\ \ [\lambda_6] = L^2,\ \text{ etc..} \tag{1.438}$$

Criterion for renormalisibility: Only terms whose parameters have zero or negative dimension are allowed in the Lagrangian. Adding any other terms will make the theory non-renormalisable. General relativity a la Einstein is the typical example of this problem: the coupling is Newton's constant whose dimension is $[G] = L^2$. Recall $S = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R$.

Correlation functions: So far the emphasise has been on the quantum properties of the free part of these theories although we have not yet discussed the more intricate aspects of the quantised Maxwell's theory. This will be done towards the end of this chapter and in the next one where gauge invariance will become crucial to understand. Quantising the free parts has given us creation and annihilation operators, quantum fields, free Hamiltonians and other charges, and Feynman propagators.

Feynman propagators like $D_F = \langle 0|T(\phi(x_2)\phi(x_1))|0\rangle$ can be understood as giving the correlation between two fields at different points and we will often refer to them as **two-point correlation functions** and sometimes just as **2-point functions**. In fact, the concept of correlation function is very general and is used in a much wider context than just particle physics. More generally we can also talk about **n-point functions** having n fields in the correlation function $\langle 0|T(\phi(x_n).....\phi(x_1))|0\rangle$.

So far we have only computed 2-point functions (propagators) in free field theories. It is very important to realise that the reason this was possible was that we could actually solve the free theory exactly in terms of their mode expansions which satisfy the free equations of motion. In order to compute such 2- or n-point functions in an interacting field theory one must in principle first find the exact expressions for the quantum fields by solving the interacting field equations. The problem with this is that interacting theories in four space-time dimensions are basically impossible to solve so one has to use a different approach, namely **perturbation theory**. In perturbation theory one starts from the free theory and adds corrections order by order in the coupling constant. These corrections are computed from the interaction terms in the Hamiltonian $\mathcal{H}_{int}$.

Comment:[36] Solvable or integrable theories are quite common in two space-time dimensions, but are a lot more rare in 2+1 dimensions and more or less non-existent in 3+1 and higher dimensions. Here *integrable* refers to theories with an infinite number of conserved charges which makes it in principle possible to solve the theory. If such theories can also be solved explicitly they are called *solvable*. All free theories are solvable as we have already demonstrated.

In a free field theory one starts by solving the field equations and then derives the appropriate Green's function, e.g., the Feynman propagator. In the interacting case one would like to do the same, namely derive the exact propagator, or *exact* two-point function, along with the ones with more external legs, the *exact n-point correlation functions*.

We will now summarise the steps used in this chapter to derive some amazing formulas for the exact n-point functions and then show how to extract scattering amplitudes and cross sections from them. We use real scalar fields in all formulas below but they are valid for any field theory.

The crucial steps, to be derived in detail in the rest of the chapter, are

1: The exact two-point function: We have so far derived the Feynman propagator only in the case of free theories. From now on will refer to the Feynman propagators as a **correlation function**, in this case the 2-point correlation function or just the 2-point function. Going from free to **interacting** quantum field theories we would like to compute the **exact 2-point function**, a step we express as follows

$$\langle 0|T\phi(x_2)\phi(x_1)|0\rangle \to \langle \Omega|T\phi^{exact}(x_2)\phi^{exact}(x_1)|\Omega\rangle. \tag{1.439}$$

This step involves finding the exact lowest energy state $|\Omega\rangle$ which will be extremely hard to find and express in any concrete terms. The energy of this state might not be zero and may even end up below zero. (Normal ordering is not an option since we don't know anything explicitly here.) The exact quantum field $\phi^{exact}(x)$, which is supposed to be an exact solution of the interacting equations of motion, is also basically impossible to find. This step should be viewed as a continuous process where we turn the coupling constant from zero in the free case to non-zero values (often > 0) for the interacting theory: $\lambda = 0 \to \lambda > 0$.

To solve interacting field theories in four space-time dimensions is generally not possible so it is extremely remarkable that one can derive a formula which expresses the exact 2-point function $\langle \Omega|T\phi^{exact}(x_2)\phi^{exact}(x_1)|\Omega\rangle$ in terms of quantities that we are familiar with from the free theory, that is $|0\rangle$ and the free field ϕ. For reasons to be explained below

[36] See PS Sect. 22.3 and Wikipedia "List of integrable models".

this requires the use of the **interaction picture** (probably familiar from QM) and hence the free fields are here denoted ϕ_I. The is provided by the **amazing formula no 1** (PS eq. 4.31) (note the over-all time ordering on the RHS)

$$\langle\Omega|T\phi^{exact}(x)\phi^{exact}(y)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\phi_I(x)\phi_I(y)e^{-i\int d^4x\mathcal{H}_I}|0\rangle}{\langle 0|e^{-i\int d^4x\mathcal{H}_I}|0\rangle}, \tag{1.440}$$

where the time integrals in the exponents are $\int_{-T}^{T} dt$ and $\mathcal{H}_I$ refers to the fact that in the Hamiltonian interaction term $\mathcal{H}_{int}(\phi)$ the field ϕ has been replaced by ϕ_I, i.e., $\mathcal{H}_I(\phi_I) := \mathcal{H}_{int}(\phi_I)$. Thus the whole RHS is expressed in terms of only familiar objects.

2: n-point functions: The 2-point function discussed above can be directly generalised to any number n of fields $\phi(x)$ as

$$\langle\Omega|T\phi^{exact}(x_n)....\phi^{exact}(x_1)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\phi_I(x_n)....\phi_I(x_1)e^{-i\int d^4x\mathcal{H}_I}|0\rangle}{\langle 0|e^{-i\int d^4x\mathcal{H}_I}|0\rangle}. \tag{1.441}$$

3. Diagram expansion: On the RHS of above formula one can expand the exponential functions in power series which can be understood as a Feynman diagram expansion. This expansion is summarised by the **amazing formula no 2** (PS eq. 4.57):

$$\begin{aligned}&\langle\Omega|T\phi^{exact}(x_n)....\phi^{exact}(x_1)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \langle 0|T\phi_I(x_n)....\phi_I(x_1)e^{-i\int d^4x\mathcal{H}_I}|0\rangle|_{connected}\\ &=\text{the sum of all connected Feynman diagrams with } n \text{ external lines.}\end{aligned} \tag{1.442}$$

This is an amazing result since the entire denominator in the previous amazing formula has now disappeared. This is due to a cancellation of all disconnected (vacuum) diagrams between the numerator and the denominator. This power series expansion is of course also a power series in the coupling constant(s).

4. S-matrix elements: Thus we have arrived at the result

$$\langle\Omega|T\phi^{exact}(x_n)....\phi^{exact}(x_1)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \langle 0|T\phi_I(x_n)....\phi_I(x_1)e^{-i\int d^4x\mathcal{H}_I}|0\rangle|_{connected}. \tag{1.443}$$

The quantum fields in the interaction picture has exactly the same mode expansion as the free fields discussed earlier in the course and hence, as seen before, ($|\mathbf{p}\rangle_0 := \sqrt{2E_{\mathbf{p}}}a^\dagger_{\mathbf{p}}|0\rangle$)

$$\phi_I(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot x}|\mathbf{p}\rangle_0. \tag{1.444}$$

To relate the above expressions for the correlation functions to scattering amplitudes one brings the points x_i to either future or past infinity in time $t \to \pm\infty$ as well as infinitely far away from the interaction region. Let us consider the 4-point function as an example. Then we concentrate on the term in the time ordered product for which t_3 and t_4 are later than t_1 and t_2 and replace all the fields in the last formula with momentum states $|\mathbf{p_1 p_2}\rangle_{in}$

and ${}_{out}\langle \mathbf{p_3 p_4}|$. Both states are to be considered as products of the individual free field momentum states $|\mathbf{p}\rangle_0$. The result is the **amazing formula no 3**

$$\langle\Omega|T\phi^{exact}(x_4)....\phi^{exact}(x_1)|\Omega\rangle \to \lim_{T\to\infty(1-i\epsilon)} {}_{out}\langle \mathbf{p_3 p_4}|Te^{-i\int d^4x\mathcal{H}_I}|\mathbf{p_1 p_2}\rangle_{in}|_{connected} \to$$
$$\lim_{T\to\infty(1-i\epsilon)} {}_0\langle \mathbf{p_3 p_4}|Te^{-i\int d^4x\mathcal{H}_I}|\mathbf{p_1 p_2}\rangle_0|_{connected,amputated,renormalised}. \tag{1.445}$$

The new words needed to define the last expression has to do with complications on the external legs of the Feynman diagrams which therefore must be cut off, i.e., **amputated**. This step will be derived in full detail later: It is associated with an infinite constant that has to be introduced into the above formula, the so called **field renormalisation**. After these steps the in and out states are just products of free field theory momentum states.

Having defined the last expression one can then get the scattering amplitude from it by

$$\lim_{T\to\infty(1-i\epsilon)} {}_0\langle \mathbf{p_3 p_4}|Te^{-i\int d^4x\mathcal{H}_I}|\mathbf{p_1 p_2}\rangle_0|_{fully\, connected,amputated,renormalised}$$
$$= (i\mathcal{M})(2\pi)^4\delta^4(p_1+p_2-p_3-p_4), \tag{1.446}$$

where **fully connected** now refers to the fact that the scattering amplitude $\mathcal{M}$ does not include trivial Feynman diagrams corresponding to particles going through the interaction region without being scattered. This corresponds to the split of the S-matrix $S = \mathbf{1} + iT$ where the matrix elements of iT are identified with $(i\mathcal{M})(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)$.

5. Cross sections: Finally, the scattering amplitude $\mathcal{M}$ obtained from the above formula is to be the used in the computation of the cross section, here presented for the scattering $2 \to 2$, i.e., for two particles A and B scattering into 2 particles $1, 2$, in the CM system (PS eq. 4.84)

$$\left(\frac{d\sigma}{d\Omega}\right)|_{CM} = \frac{1}{2E_A 2E_B|v_A - v_B|}\frac{|\mathbf{p}_1|}{(2\pi)^2 4E_{CM}}|\mathcal{M}(p_A p_B \to p_1, p_2)|^2. \tag{1.447}$$

Note: Once the Feynman rules are understood it is often a rather straightforward matter to apply them to any scattering process at tree level and compute the cross section from the last formula above (or a similar one if there are many particles in the out-state). For loop (radiative) corrections and renormalisation we need some additional technology that we will develop after we have done the tree diagram calculations in the chapter 5.

1.11.2 Correlation functions and perturbation theory

In the discussion below we will use ϕ^4 theory for simplicity but the arguments are valid for any kind of interacting field theories. So let's recall the Lagrangian for the ϕ^4 theory

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}(\lambda) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{1.448}$$

and the Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{int}(\lambda) = \frac{1}{2}\Pi^2 + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4, \tag{1.449}$$

where the interaction terms are all terms with more than two fields, here just $\mathcal{L}_{int}(\lambda) = -\frac{\lambda}{4!}\phi^4$ or for the Hamiltonian $\mathcal{H}_{int}(\lambda) = \frac{\lambda}{4!}\phi^4$. We will consider the *coupling constant* λ to be a parameter that we can dial up and down, and thus continuously turn a free theory (for $\lambda = 0$) into an interacting one with $\lambda > 0$.

It is then important to realise that the states $|0\rangle$, $a^\dagger_{\mathbf{p}}|0\rangle$, $a^\dagger_{\mathbf{p_2}}a^\dagger_{\mathbf{p_1}}|0\rangle$ etc in the free theory have energies 0, $E_{\mathbf{p}}$, $E_{\mathbf{p_1}}+E_{\mathbf{p_2}}$ etc and that both the states and the energies will depend on λ when $\lambda > 0$. The states becomes extremely complicated since we cannot in general solve the theory exactly and the energy eigenvalues of the interacting Hamiltonian will change away from their free values. The energy of the exact ground states $|\Omega\rangle$ might even move below zero. However, for any value of λ the set of all energy eigenstates will continue to be a complete set of states.

The issue at this point is therefore: If we cannot solve the interacting theory and find the exact energy eigenstates $|\Omega\rangle_{(\lambda)}$, $|\mathbf{p}\rangle_{(\lambda)}$, $|\mathbf{p_2}\mathbf{p_1}\rangle_{(\lambda)}$ etc and the exact expressions for the quantum fields $\phi^{exact}_{(\lambda)}(x)$ what can we do? The goal might be, if possible, to try to express the exact correlation functions like the 2-point function

$${}_{(\lambda)}\langle\Omega|T\phi^{exact}_{(\lambda)}(x)\phi^{exact}_{(\lambda)}(y)|\Omega\rangle_{(\lambda)}, \tag{1.450}$$

in terms of quantities we are familiar with like the free $\lambda = 0$ vacuum $|0\rangle$, the free one- and multi-particle states $|\mathbf{p}\rangle_0$, $|\mathbf{p_2}\mathbf{p_1}\rangle = |\mathbf{p_2}\rangle_0|\mathbf{p_1}\rangle_0$ etc, and the free quantum field

$$\phi(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(a_{\mathbf{p}}e^{-ip\cdot x} + a^\dagger_{\mathbf{p}}e^{ip\cdot x}). \tag{1.451}$$

Amazingly the answer is that this can be done as we will see now when we start to develop **perturbation theory**.

For the free theory the Feynman propagator is the exact 2-point function[37]:

$$D_F(x_2-x_1) := \langle 0|T\phi(x_2)\phi(x_1)|0\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip(x_2-x_1)} = D(x_2-x_1)+D(x_1-x_2), \tag{1.452}$$

[37] If we let the time ordering be applied to all the fields in the correlation function the bracket in $\langle 0|T(....)|0\rangle$ is not needed!

where the T refers to the fact that the correlation function in question is *time ordered* and

$$D(x_2 - x_1) = \langle 0|\phi(x_2)\phi(x_1)|0\rangle = \theta(t_2 - t_1)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip(x_2-x_1)}. \tag{1.453}$$

In the interacting case we want to find the exact 2-point function which takes the form

$${}_{(\lambda)}\langle\Omega|T\phi^{exact}_{(\lambda)}(x)\phi^{exact}_{(\lambda)}(y)|\Omega\rangle_{(\lambda)}. \tag{1.454}$$

To follow the notation of the book we now drop the index λ and the *exact* on these quantities and write this correlation function as

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle, \tag{1.455}$$

where we now have to keep in mind what things mean: $|\Omega\rangle$ is the **exact ground state** and $\phi(x)$ denotes the **exact quantum field**. We will soon introduce another quantum field $\phi_I(x)$, the *interaction picture field*, which at the end will also be denoted $\phi(x)$. Once the formalism has been developed and is ready to be applied to physics problems this rather messy set of notations will not be a problem!

The first step towards this goal is to recall that when the classical field theory is quantised we end up in the Heisenberg picture where operators, like the exact quantum field $\phi(x)$, depends on time and states are time-independent. This fact means that

$$\phi(x) = \phi(t, \mathbf{r}) = e^{iHt}\phi(0, \mathbf{r})e^{-iHt}, \tag{1.456}$$

which is here an exact statement generalising the free field theory result obtained earlier having H_0 in the exponents. In the free case we, of course, have the mode expansion of $\phi_{free}(x)$ for all t which is not the case at all for the exact $\phi(x)$. However, nothing can stop us from at one particular time, say $t = 0$, expand the exact field in modes, i.e., we have for the above field at $t = 0$ (sometimes denoted ϕ_0)

$$\phi(0, \mathbf{r}) := \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{r}} + a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{r}}) \tag{1.457}$$

The second step is then a crucial one: instead of taking this over to the Schrödinger picture by defining the S-picture state $|\psi\rangle_S$ from the H-picture state $|\psi\rangle_H$,

$$e^{iHt}\phi(0, \mathbf{r})e^{-iHt}|\psi\rangle_H := e^{iHt}\phi(0, \mathbf{r})|\psi(t)\rangle_S, \tag{1.458}$$

using the full Hamiltonian H leaving S-picture operators like $\phi_S(0, \mathbf{r}) = \phi(0, \mathbf{r})$ time-independent, we define the **interaction picture** by defining

$$\phi_I(x) = \phi_I(t, \mathbf{r}) = e^{iH_0t}\phi(0, \mathbf{r})e^{-iH_0t}, \tag{1.459}$$

which then at the same time gives states $|\psi(t)\rangle_I$ a time dependence essentially generated by H_{int}. The details will be presented below. Note that if we compute ϕ_I it will at all times t have the same form as the free fields analysed in the beginning of this course, in particular the same mode expansion. In fact, ϕ_I is just a free field so we have achieved one of the goals mentioned above. The next point is therefore to write the whole interacting theory in terms of this field ϕ_I.

To achieve this goal we must first express the exact field $\phi(x)$ in terms of ϕ_I. This is easily done by combining the equations above:

$$\phi(x) = \phi(t, \mathbf{r}) = (e^{iHt}e^{-iH_0t})\phi_I(t, \mathbf{r})(e^{iH_0t}e^{-iHt}) := U^\dagger(t)\phi_I(x)U(t). \tag{1.460}$$

Here we have also defined the unitary time evolution operator $U(t) := e^{iH_0t}e^{-iHt}$ which seems to depend only on H_{int} but this is not quite true since H and H_0 do not in general commute which means that the two exponents cannot be put together in one exponent. Note that the whole Hamiltonian H is time-independent and hence expressed in terms of $\phi(0, \mathbf{r})$!

As we will now show it is possible to express also $U(t)$ in terms the interaction picture field ϕ_I. To this end we reintroduce a general time t_0 (which was equal to zero above):

$$U(t, t_0) := e^{iH_0(t-t_0)}e^{-iH(t-t_0)}. \tag{1.461}$$

The trick to see that this operator is really a function of $\phi_I(x)$ is to first show that it is a solution of the Schrödinger equation. By computing its time-derivative we get

$$i\partial_t U(t, t_0) = e^{iH_0(t-t_0)}(-H_0)e^{-iH(t-t_0)} + e^{iH_0(t-t_0)}(H)e^{-iH(t-t_0)} = e^{iH_0(t-t_0)}(H_{int})e^{-iH(t-t_0)}. \tag{1.462}$$

The key observation now is that, e.g., in ϕ^4 theory we have $H_{int} = \frac{\lambda}{4!}\int d^3r\phi^4(0, \mathbf{r})$ and hence we can replace this field by $\phi_I(x)$ using (see the inverse equation above)

$$\phi(0, \mathbf{r}) = e^{-iH_0(t-t_0)}\phi_I(x)e^{iH_0(t-t_0)}, \tag{1.463}$$

which implies that the Schrödinger equation above becomes

$$i\partial_t U(t, t_0) = H_I(t)U(t, t_0), \text{ where } H_I := H_{int}(\phi_I). \tag{1.464}$$

Thus solving this equation for $U(t, t_0)$ must mean that also this operator becomes expressed entirely in terms of $\phi_I(x)$!

Solving this Schrödinger equation requires however some careful steps. Let us try a power series, satisfying $U(t_0, t_0) = 1$, called a Dyson series,

$$U(t, t_0) = 1 + (-i)\int_{t_0}^{t} dt_1 H_I(t_1) + (-i)^2\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2) + ... \tag{1.465}$$

This series has some interesting features:
1. $t > t_1 > t_2 > ... > t_0$.
2. The operators $H_I(t_1)H_I(t_2)$... are thus time-ordered with operators (non-commuting) at earlier times to the right of the later ones.
3. There are no factors $1/n!$ so this is not the expansion of an exponential function.

4. It solves the Schrödinger equation trivially!
5. It can be written as an exponential with a T-ordered exponent due to the following fact:

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2) = \frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 T(H_I(t_1)H_I(t_2)) \tag{1.466}$$

which generalises to all higher terms with n H_I-operators producing factors $1/n!$. Note that the integration region on the LHS is a triangle in t_1, t_2 while it is a square on the RHS.

The nice result is therefore that the time-evolution operator can be written as a time ordered exponential

$$U(t,t_0) = T\left(e^{-i\int_{t_0}^{t} dt' H_I(t')}\right). \tag{1.467}$$

Expressing the above definition of $U(t,t_0)$ slightly differently as (satifies the same Schoedinger equation as above and equals 1 for $t = t_0$)

$$U(t,t_0) = e^{iH_0 t}e^{-iH(t-t_0)}e^{-iH_0 t_0}, \tag{1.468}$$

implies some very useful identities

$$U(t,t')U(t',t_0) = U(t,t_0), \;\; (U(t,t_0))^{\dagger} = (U(t,t_0))^{-1} = U(t_0,t). \tag{1.469}$$

We have come a long way towards the goal of expressing everything in terms of quantities from the free theory but we still need to discuss the vacuum state $|\Omega\rangle$ to see if it can be expressed in terms of the vacuum $|0\rangle$ of the free theory. This can be done but it requires some assumptions: consider the full Hamiltonian acting on the free vacuum

$$e^{-iHT}|0\rangle = \tilde{\Sigma}_{n=0}^{\infty} e^{-iE_n T}|n\rangle\langle n|0\rangle = e^{-iE_0 T}|\Omega\rangle\langle\Omega|0\rangle + \tilde{\Sigma}_{n=1}^{\infty} e^{-iE_n T}|n\rangle\langle n|0\rangle, \tag{1.470}$$

where we have inserted a complete set of exact energy eigenstates $|n\rangle$ without knowing exactly what they are (the tilde summation indicates that part of the spectrum might be continuous and thus requires an integral). In the last expression we have used the definition $|n=0\rangle := |\Omega\rangle$ and assumed that $\langle\Omega|0\rangle \neq 0$. The last terms in this expression can be driven to zero by letting $T \to \infty(1-i\epsilon)$ assuming that all energies (except perhaps E_0) stay positive when λ is turned on. Then we can solve for $|\Omega\rangle$:

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{e^{-iHT}|0\rangle}{e^{-iE_0 T}\langle\Omega|0\rangle} = \lim_{T\to\infty(1-i\epsilon)} \frac{U(0,-T)|0\rangle}{e^{-iE_0 T}\langle\Omega|0\rangle}, \tag{1.471}$$

where we to get the last form of the answer used that $e^{-iHT}|0\rangle = e^{-iHT}e^{iH_0 T}|0\rangle$ which is OK since $H_0|0\rangle = 0$. Similarly for $\langle\Omega|$

$$\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T,0)}{e^{-iE_0 T}\langle 0|\Omega\rangle}. \tag{1.472}$$

Putting together all these results we end up with a truly amazing formula for the exact correlation function: consider first the case $t_2 > t_1$

$$\langle\Omega|\phi(x_2)\phi(x_1)|\Omega\rangle|_{t_2>t_1} = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T,0)U^{\dagger}(t_2,0)\phi_I(x_2)U(t_2,0)U^{\dagger}(t_1,0)\phi_I(x_1)U(t_1,0)U(0,-T)|0\rangle}{e^{-iE_0 2T}|\langle\Omega|0\rangle|^2}. \tag{1.473}$$

Using the identities established above for the operator $U(t, t_0)$ this result for $t_2 > t_1$ can be simplified to

$$\langle\Omega|\phi(x_2)\phi(x_1)|\Omega\rangle|_{t_2>t_1} = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T,t_2)\phi_I(x_2)U(t_2,t_1)\phi_I(x_1)U(t_1,-T)|0\rangle}{e^{-iE_0 2T}|\langle\Omega|0\rangle|^2}. \tag{1.474}$$

Having obtained this formula we can use it to get a simpler expression for the denominator. Removing the two fields gives (the ground state is always assumed to be normalised to 1)

$$1 = \langle\Omega|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T,t_2)U(t_2,t_1)U(t_1,-T)|0\rangle}{e^{-iE_0 2T}|\langle\Omega|0\rangle|^2} = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T,-T)|0\rangle}{e^{-iE_0 2T}|\langle\Omega|0\rangle|^2}, \tag{1.475}$$

which gives the denominator above in terms of the time evolution operator.

The final step is to add up all possible time orderings of the $\phi(x)$ fields and generalise from 2 to n operators. This gives the final form of the n-point function which is still an exact expression:

$$\langle\Omega|T\phi(x_n)....\phi(x_1)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\left(\phi_I(x_n)....\phi_I(x_1)U(T,-T)\right)|0\rangle}{\langle 0|U(T,-T)|0\rangle}. \tag{1.476}$$

Perturbation theory can now be obtained as a power series in the Hamiltonian interaction terms if we replace the time evolution operator in the above formula with its explicit form in terms of $\mathcal{H}_I$. This gives the following equation expressing the exact n-point function in terms of the interaction picture quantum field $\phi_I(x)$ and the interaction part of the Hamiltonian $\mathcal{H}_I$:

$$\langle\Omega|T\phi(x_n)....\phi(x_1)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\phi_I(x_n).....\phi_I(x_1)e^{-i\int d^4x\mathcal{H}_I}|0\rangle}{\langle 0|e^{-i\int d^4x\mathcal{H}_I}|0\rangle}, \tag{1.477}$$

where the interaction part of the Hamiltonian, $\mathcal{H}_I(\phi_I)$, is the original set of interaction terms, $\mathcal{H}_{int}(\phi)$ in the Hamiltonian expressed in terms of the time independent ϕ_0 replaced by the time dependent interaction picture field $\phi_I(x)$ whose time dependence is due only to the free part of the Hamiltonian. Note that on the RHS the time integral in the exponent goes from $-\infty$ to $+\infty$ in a slightly imaginary direction given by $T(1-i\epsilon)$ as T goes to ∞.

Comment: The above formula for $\langle\Omega|T\phi(x_n)...\phi(x_1)|\Omega\rangle$ can be used in any field theory and is in some sense still exact. By expanding the exponential one obtains an infinite set of terms that can be used to compute the correlation function to any degree of accuracy. However, often such a perturbation series start to diverge after a certain number of terms as a function of the coupling constant (provided it is positive and smaller than 1) which means the series instead constitutes a so called *asymptotic series*. Such series can fortunately be subjected to a Borel resummation and be given a proper definition in terms of a special kind of non-perturbative function. This function will contain terms that come from instanton solutions of the field equations in the way discussed in the introductory lecture. The modern area in mathematics dealing with such phenomena is called *resurgence* and has developed into a very important tool in physics in the last few years. The new features appearing in these functions are called *non-perturbative* since they can be studies by completely different methods, e.g. as classical solutions of the field equations, related to instantons, monopoles etc. These aspects will not be discussed further in this course.