1.11.3 Wick's theorem

Let us return to the amazing formula derived above

$$\langle\Omega|T\phi(x_n)....\phi(x_1)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\phi_I(x_n).....\phi_I(x_1)e^{-i\int d^4z\mathcal{H}_I}|0\rangle}{\langle 0|e^{-i\int d^4z\mathcal{H}_I}|0\rangle}, \tag{1.478}$$

and discuss how to use it in perturbation theory. The next task will thus be to develop efficient methods which can be used to compute term by term in the expansion of the exponential. The first step in this process is to learn how to compute correlation functions of an arbitrary number of time-ordered quantum fields located at arbitrary spacetime points. Note that these points can even be the same if a subset of the fields come from the same interaction term $\int d^4z\mathcal{H}_I(\phi_I(z)) = \frac{\lambda}{4!}\int d^4z(\phi_I(z))^4$.

Thus we need to compute correlation functions like

$$\langle 0|T\phi(x_n)...\phi(x_1)|0\rangle. \tag{1.479}$$

Note: we will from now on drop the index I on the interaction picture fields ϕ_I which have appeared in all previous formulas.

The main trick is provided by *Wick's theorem* which expresses a time-ordered product of interaction picture quantum fields in terms of normal-ordered products of the fields. The first step is to split each quantum field into an annihilation and a creation part as follows:

$$\phi(x) = \phi^+(x) + \phi^-(x), \tag{1.480}$$

where the two parts are defined by

$$\phi^+(x)|0\rangle = 0, \;\; \langle 0|\phi^-(x) = 0, \tag{1.481}$$

that is, all annihilation operator terms in $\phi(x)$ appear in $\phi^+(x)$ while all creation operator terms are collected in $\phi^-(x)$:

$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}a_{\mathbf{p}}e^{-ip\cdot x}, \;\; \phi^-(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}a^\dagger_{\mathbf{p}}e^{ip\cdot x}. \tag{1.482}$$

In this language normal-ordering, denoted as usual by : .. :, means that any product of fields when normal-ordered is written with all $-$ parts of $\phi(x)$ to the left of the $+$ parts so that the normal-ordered expression has zero vacuum expectation value. As an example consider

$$\begin{aligned} &:\phi(x)\phi(y) :=: (\phi^+(x) + \phi^-(x))(\phi^+(y) + \phi^-(y)) : \\ &= \phi^+(x)\phi^+(y) + \phi^-(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \phi^-(x)\phi^-(y), \end{aligned} \tag{1.483}$$

where in the third term we have flipped the order of the two operators *without* adding the commutator. This is the content of normal ordering. Note that for the first and last terms the order of the fields does not matter. Now $\langle 0| : \phi(x)\phi(y) : |0\rangle = 0$ is trivially true.

However, in order to relate such normal-ordered expressions to the time-ordered ones these commutators are precisely the point. For the 2-point correlation function this is done as follows

$$\begin{aligned} T\phi(x)\phi(y) = \theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x) \\ =: \phi(x)\phi(y) : +\theta(x^0 - y^0)[\phi^+(x), \phi^-(y)] + \theta(y^0 - x^0)[\phi^+(y), \phi^-(x)]. \end{aligned} \tag{1.484}$$

To get this result we have used, in the term with $x^0 > y^0$,

$$\phi^+(x)\phi^+(y) + \phi^-(x)\phi^+(y) + \phi^+(x)\phi^-(y) + \phi^-(x)\phi^-(y) =: \phi(x)\phi(y) : +[\phi^+(x), \phi^-(y)], \tag{1.485}$$

since only the third term needs to be rewritten. Similarly for the term with $y^0 > x^0$, we get

$$\phi^+(y)\phi^+(x) + \phi^+(y)\phi^-(x) + \phi^-(y)\phi^+(x) + \phi^-(y)\phi^-(x) =: \phi(y)\phi(x) : +[\phi^+(y), \phi^-(x)]. \tag{1.486}$$

The sum of the two commutator terms is called the *contraction* which is denoted as follows

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : +\contraction{}{\phi}{(x)}{\phi}\phi(x)\phi(y). \tag{1.487}$$

However, by doing the commutators, e.g.,

$$\begin{aligned} [\phi^+(x), \phi^-(y)] &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} e^{-ip\cdot x} \int \frac{d^4p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}'}}} e^{ip'\cdot y} [a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot(x-y)} := D(x-y), \end{aligned} \tag{1.488}$$

and the same calculation for the other commutator gives $D(y-x)$. Thus we see that the contractions are no longer operators and, in fact, exactly reproduce the two terms $D(x-y)$ and $D(y-x)$ making up the Feynman propagator $D_F(x-y)$. Therefore, by sandwiching the above relation between vacuum states, we get the following well-known relation

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \langle 0| : \phi(x)\phi(y) : |0\rangle + \langle 0|\contraction{}{\phi}{(x)}{\phi}\phi(x)\phi(y)|0\rangle = D_F(x-y), \tag{1.489}$$

since the first term is zero due to the normal-ordering and in the second term the contraction in not an operator so we can use $\langle 0|0\rangle = 1$. Turning this argument around, since this equation must be true it implies that the contraction is just the Feynman propagator.

By repeating this exercise for four fields we find , denoting $\phi(x_n)$ as just ϕ_n,

$$T\phi_1\phi_2\phi_3\phi_4 = (\phi_1^+ + \phi_1^-)(\phi_2^+ + \phi_2^-)(\phi_3^+ + \phi_3^-)(\phi_4^+ + \phi_4^-)$$

$$= \phi_1^+\phi_2^+\phi_3^+\phi_4^+ + \phi_1^-\phi_2^+\phi_3^+\phi_4^+ + \phi_1^+\phi_2^-\phi_3^+\phi_4^+ + \phi_1^+\phi_2^+\phi_3^-\phi_4^+ + \phi_1^+\phi_2^+\phi_3^+\phi_4^- +$$

$$\phi_1^-\phi_2^-\phi_3^+\phi_4^+ + \phi_1^-\phi_2^+\phi_3^-\phi_4^+ + \phi_1^-\phi_2^+\phi_3^+\phi_4^- + \phi_1^+\phi_2^-\phi_3^-\phi_4^+ + \phi_1^+\phi_2^-\phi_3^+\phi_4^- + \phi_1^+\phi_2^+\phi_3^-\phi_4^- +$$

$$\phi_1^-\phi_2^-\phi_3^-\phi_4^+ + \phi_1^-\phi_2^-\phi_3^+\phi_4^- + +\phi_1^-\phi_2^+\phi_3^-\phi_4^- + \phi_1^+\phi_2^-\phi_3^-\phi_4^- + \phi_1^-\phi_2^-\phi_3^-\phi_4^-$$

$$
\begin{aligned}
&=: \phi_1\phi_2\phi_3\phi_4 : + : \wick{\c1\phi_1 \c1\phi_2}\phi_3\phi_4 : + : \wick{\c1\phi_1 \phi_2 \c1\phi_3}\phi_4 : + : \wick{\c1\phi_1 \phi_2 \phi_3 \c1\phi_4} : \\
&\quad : \phi_1\wick{\c1\phi_2 \c1\phi_3}\phi_4 : + : \phi_1\wick{\c1\phi_2 \phi_3 \c1\phi_4} : + : \phi_1\phi_2\wick{\c1\phi_3 \c1\phi_4} : \\
&\quad + \wick{\c1\phi_1 \c1\phi_2 \c2\phi_3 \c2\phi_4} + \wick{\c1\phi_1 \c2\phi_2 \c1\phi_3 \c2\phi_4} + \wick{\c1\phi_1 \c2\phi_2 \c2\phi_3 \c1\phi_4}.
\end{aligned} \tag{1.490}
$$

Here we have not considered the different time-ordered terms separately but just noted that when they are added together they give rise to contractions which, as we saw above, contain the two possible time-orderings of the two fields in the contraction: $\wick{\c1\phi_1 \c1\phi_2} = \theta(x_1^0 - x_2^0)[\phi_1^+, \phi_2^-] + \theta(x_2^0 - x_1^0)[\phi_2^+, \phi_1^-]$.

The easiest way to pick up all the commutators in the expression above is to move, step by step, all the ϕ^- to the far left of all the ϕ^+ in each term as follows: consider the 4th of the 16 terms above

$$
\begin{aligned}
\phi_1^+\phi_2^+\phi_3^-\phi_4^+ &= \phi_1^+\phi_3^-\phi_2^+\phi_4^+ + \phi_1^+[\phi_2^+, \phi_3^-]\phi_4^+ = \phi_3^-\phi_1^+\phi_2^+\phi_4^+ + [\phi_1^+, \phi_3^-]\phi_2^+\phi_4^+ + \phi_1^+[\phi_2^+, \phi_3^-]\phi_4^+ \\
&= \phi_3^-\phi_1^+\phi_2^+\phi_4^+ + [\phi_1^+, \phi_3^-]\phi_2^+\phi_4^+ + [\phi_2^+, \phi_3^-]\phi_1^+\phi_4^+,
\end{aligned} \tag{1.491}
$$

where the last step is possible since the result of doing the commutator is not an operator and can hence be moved to the left of any operator fields.

The above example shows that the contracted fields can be moved outside the normal ordering. For example, $: \wick{\c1\phi_1 \phi_2 \c1\phi_3}\phi_4 := D_F(x_1 - x_3) : \phi_2\phi_4 :$. The terms that are contracted twice are, in this example with four fields, not operators.

Terms with double contractions arise as follows. Consider the 11th of the 16 terms above:

$$
\begin{aligned}
\phi_1^+\phi_2^+\phi_3^-\phi_4^- &= \phi_3^-\phi_4^-\phi_1^+\phi_2^+ + [\phi_1^+\phi_2^+, \phi_3^-\phi_4^-] = \phi_3^-\phi_4^-\phi_1^+\phi_2^+ + \phi_1^+[\phi_2^+, \phi_3^-\phi_4^-] + [\phi_1^+, \phi_3^-\phi_4^-]\phi_2^+ \\
&= \phi_3^-\phi_4^-\phi_1^+\phi_2^+ + \phi_1^+\phi_3^-[\phi_2^+, \phi_4^-] + \phi_1^+[\phi_2^+, \phi_3^-]\phi_4^- + \phi_3^-[\phi_1^+, \phi_4^-]\phi_2^+ + [\phi_1^+, \phi_3^-]\phi_4^-\phi_2^+,
\end{aligned} \tag{1.492}
$$

where the second and third terms after one contraction are still not normal-ordered. So to get a fully normal-ordered expression another step is needed and a new commutator will arise. The expression above then becomes

$$
\begin{aligned}
&= \phi_3^-\phi_4^-\phi_1^+\phi_2^+ + \phi_3^-\phi_1^+[\phi_2^+, \phi_4^-] + [\phi_1^+, \phi_3^-][\phi_2^+, \phi_4^-] + \phi_4^-\phi_1^+[\phi_2^+, \phi_3^-] + [\phi_1^+, \phi_4^-][\phi_2^+, \phi_3^-] \\
&\quad + \phi_3^-\phi_2^+[\phi_1^+, \phi_4^-] + \phi_4^-\phi_2^+[\phi_1^+, \phi_3^-].
\end{aligned} \tag{1.493}
$$

The following theorem generalises this to an arbitrary number of fields:

$$
\text{Wick's theorem :} \quad T\phi_1 ... \phi_n =: \phi_1 ... \phi_n : + \textit{ all possible contractions.} \tag{1.494}
$$

The proof is most easily done by induction.

For any n-point correlation function it is only the terms which are fully contracted (i.e., terms that have no normal-ordered factor in them) that contribute. The 4-point function considered above thus becomes:

$$\langle 0|T\phi_1\phi_2\phi_3\phi_4|0\rangle = D_F(x_1 - x_2)D_F(x_3 - x_4)$$

$$+ D_F(x_1 - x_3)D_F(x_2 - x_4) + D_F(x_1 - x_4)D_F(x_2 - x_3). \qquad (1.495)$$

These three terms correspond to the three possible ways to connect four points with two lines. It will be very convenient to draw lines representing the propagators as we will start doing in the next section.

1.11.4 Feynman diagrams

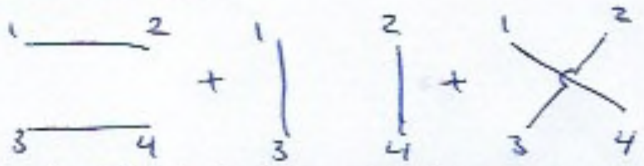
The next task is to analyse what happens when interaction terms from the exponent $e^{-i\int d^4x \mathcal{H}_I}$ are involved in the correlation function. A nice way to organise this discussion is by introducing *Feynman diagrams*. In the previous example with four non-interacting fields we found a result that implies that the four-point correlator is the sum of three terms

$$\langle 0|T\phi_1\phi_2\phi_3\phi_4|0\rangle = D_F(x_1 - x_2)D_F(x_3 - x_4)$$
$$+ D_F(x_1 - x_3)D_F(x_2 - x_4) + D_F(x_1 - x_4)D_F(x_2 - x_3). \qquad (1.496)$$

Our first **Feynman rule** is to associate a line between x_i and x_j for each real scalar field Feynman propagator $D_F(x_i - x_j)$ =

$\langle 0|T\phi(x_i)\,\phi(x_j)|0\rangle =$ 

where the line itself has no arrow on it since there is no direction involved (no charge moving along it). Later we will indicate the direction of the momentum by an arrow next to the line but this is not needed at this point. Having introduced this way to depict a propagator the 4-point function above is drawn as follows

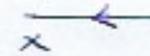
$\langle 0|T\phi_1\phi_2\phi_3\phi_4|0\rangle =$ 

As done here, the four external points x_i, $i = 1, 2, 3, 4$, should be put at the same place in all the diagrams.

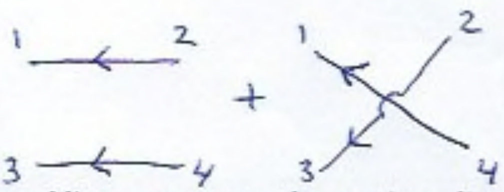
For *complex scalar fields* there is a direction of charge-flow along the two lines which therefore cannot connect the four points in the same way as for real fields. Instead we get

$$\langle 0|T\Phi_1\Phi_2^\dagger\Phi_3\Phi_4^\dagger|0\rangle = D_F(x_1 - x_2)D_F(x_3 - x_4) + D_F(x_1 - x_4)D_F(x_3 - x_2), \qquad (1.497)$$

where the Feynman propagator now is, and hence requires an arrow from the y end to the x end of the line, (recall that $\bar{\Phi}$ contains $a_{\mathbf{p}}^\dagger$)

$$D_F(x - y) := \langle 0|T\Phi(x)\bar{\Phi}^\dagger(y)|0\rangle = \qquad (1.498)$$ 

This is drawn as follows:

$\langle 0|T\Phi_1\Phi_2^+\Phi_3\Phi_4^+|0\rangle =$ 

Things get a lot more interesting when adding a vertex to the previous discussion coming from the first non-trivial term in the expansion of the exponential $e^{-i\int d^4x \mathcal{H}_I}$.

Consider the 2-point function in real scalar ϕ^4 theory. To lowest order this is just the free Feynman propagator D_F whose first correction is at order λ. This correction reads:

$$\langle 0|T\phi(x)\phi(y)(-i\int d^4z\mathcal{H}_I(z))|0\rangle = -\frac{i\lambda}{4!}\int d^4z\langle 0|T\phi(x)\phi(y)(\phi(z))^4|0\rangle. \quad (1.499)$$

Since there are now six fields in this expression it will give rise to three D_F Feynman propagators connecting the points x, y and the four z-points in all possible ways (remember "all possible contractions" in Wick's theorem). Thus we get

$$\langle 0|T\phi(x)\phi(y)(\phi(z))^4|0\rangle = \quad (1.500)$$

$= \phi(x)\ \phi(y)\ \phi(z)\ \phi(z)\ \phi(z)\ \phi(z)$

$+$

$+$ } 3 identical terms

$+ \ \phi(x)\ \phi(y)\ \phi(z)\ \phi(z)\ \phi(z)\ \phi(z)$

$+$

$+$

$\vdots$ } 12 identical terms

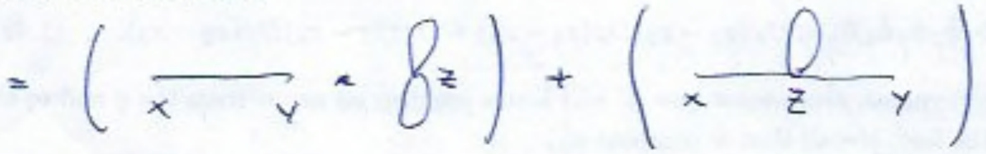
Thus the result for the correlation function is (since the four $\phi(z)$ fields are identical)

$$-\frac{i\lambda}{4!}\int d^4z(3\phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z) + 12\phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z)) \quad (1.501)$$

which can summarised as follows by using the Feynman rules

$$\langle 0|T\phi(x)\phi(y)(\phi(z))^4|0\rangle = D_F(x-y)(D_F(z-z))^2 + D_F(x-z)D_F(z-y)D_F(z-z), \quad (1.502)$$

and is drawn as follows



Here the first term corresponds to a direct propagator between x and y times the "figure-eight" two-loop diagram, and thus consists of two **disconnected** pieces, while the second term is known as the "snail diagram" and is a **connected** one-loop diagram. Parts of a diagram (or a whole diagram) which have no external legs are called vacuum diagrams or **vacuum bubbles**.

The second Feynman rule to be introduced here is the one for the vertex: (with no factor 1/4!)

 $= (-i\lambda)\int d^4z. \quad (1.503)$

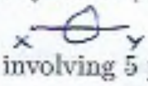
The third rule concerns the end of the external legs which, in this case of real scalar fields, is just

$$= 1. \tag{1.504}$$

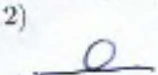
For spinors and vector fields this rule will involve polarisation spinor/tensors as we have seen indications of already.

Finally, the fourth rule has to do with the coefficients 3 and 12 that appeared in the formula above. Combining these two numbers with the factor $1/4!$ form $\mathcal{H}_I$ we get the number $3/4! = 1/8$ and $12/4! = 1/2$. Note that the Feynman rule for the vertex above did not include the factor $1/4!$ and the reason for that is seen here. By combining the factor from the vertex with the combinatorial numbers from Wick's theorem we get numbers $1/s$ where s is called the **symmetry factor**. These symmetry factors can be obtained directly by looking at the symmetries of the Feynman diagrams without going through the counting of contractions as we did above.

The figure-eight diagram has $s = 8$ because $s = 2 \times 2 \times 2$ where the factors of 2 come from:
1) The two loops are identical which means an over-counting by a factor 2 (thus we must divide by 2).
2) Each loop is over-counted by a factor 2 for real scalar fields since we treated the 4 ϕ fields from the vertex as different in the contractions. This corresponds to having lines with direction. For real scalar fields there is no direction on the lines which explains the over-counting.
The snail-diagram has $s = 2$ which again comes from the loop without a direction.

Exercise. Get the symmetry factor(s) for
1) The sun-set diagram. 
2) The different diagrams involving 5 propagators at order λ^2 in ϕ^4 (a bit lengthy but very useful exercise).

The next step in this development is to express the Feynman rules in momentum space. The result is a set of rules that are really quite easy to apply in actual amplitude calculations as we will see later. A good way to get a feeling for where these momentum space rules come from is to consider an example obtained from the Feynman rules in x-space given above. So let us consider the snail-diagram diagram (having symmetry factor $s = 2$)



$$= \frac{1}{2}(-i\lambda) \int d^4z D_F(x-z) D_F(z-y) D_F(z-z). \tag{1.505}$$

Now we can insert the integral expression for each of the Feynman propagators, i.e.,

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}, \tag{1.506}$$

giving, with momenta p_i on the legs and momentum q in the loop,

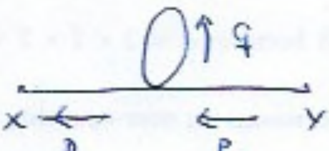
$$= \frac{1}{2}(-i\lambda)\int d^4z \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}$$

$$\frac{i}{p_1^2-m^2+i\epsilon}e^{-ip_1\cdot(x-z)}\frac{i}{p^2-m^2+i\epsilon}e^{-ip_2\cdot(z-y)}\frac{i}{q^2-m^2+i\epsilon}e^{-iq\cdot(z-z)}. \tag{1.507}$$

The last factor has a zero exponent so the z-integral gives a delta function relating the two momenta on the external legs and we get

$$= \frac{1}{2}(-i\lambda)\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}(2\pi)^4\delta^4(p_1-p_2)\frac{i^3e^{-ip_1\cdot x}e^{ip_2\cdot y}}{(p_1^2-m^2+i\epsilon)(p_2^2-m^2+i\epsilon)(q^2-m^2+i\epsilon)}.$$

where the direction of the momentum in leg 2 comes from choosing to write the propagator between y and z as $D_F(z-y)$ (note the order of the points). The diagram can then be drawn as follows



The expression for this diagram can then be obtained directly from the following rules:

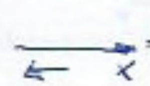
1. Real scalar propagator

$$= \frac{i}{p^2-m^2+i\epsilon}, \tag{1.508}$$

2. ϕ^4 vertex

$$= -i\lambda, \tag{1.509}$$

3. External legs

$= e^{-ip\cdot x}$,


$= e^{+ip\cdot x}$ (1.510)

4. Momentum conservation

$\Sigma_i p_i = 0$ at each vertex (from $\int d^4z$) implies over-all momentum conservation $(2\pi)^4\delta^4(\Sigma p_{final} - \Sigma p_{initial})$ (1.511)

5. For each (undetermined) momentum

$$\int \frac{d^4q}{(2\pi)^4} \tag{1.512}$$

6. Symmetry factors obtained from the symmetries of the Feynman diagram.

Note: Later when these rules are used in the context of scattering amplitudes (with in and out momentum states) they are simplified further. The momentum integrals for the external legs as well as their propagators and exponential momentum factors are dropped.

Bubble cancellation

Let us now return to the amazing formula for the exact n-point correlation function

$$\langle\Omega|T\phi^{exact}(x_n)...\phi^{exact}(x_1)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\phi(x_n)...\phi(x_1)e^{-i\int d^4z\mathcal{H}_I}|0\rangle}{\langle 0|e^{-i\int d^4z\mathcal{H}_I}|0\rangle}. \tag{1.513}$$

The discussion so far has concerned only the numerator on the RHS of this formula. When we now address the denominator we can apply the same Feynman rules as above even though there are no external legs in this case. Thus the diagram expansion of the denominator will contain only vacuum bubble diagrams. These are however all infinite. Consider the figur-eight diagram, here called V_∞, we can try to compute it from the Feynman rules as follows:

$$V_\infty = (-i\lambda)\int_{-T}^{T}dz^0\int_V d^3z\int\frac{d^4q_1}{(2\pi)^4}\int\frac{d^4q_2}{(2\pi)^4}\frac{i}{q_1^2-m^2+i\epsilon}e^{-iq_1\cdot(z-z)}\frac{i}{q_2^2-m^2+i\epsilon}e^{-iq_2\cdot(z-z)}, \tag{1.514}$$

where the space integral is over the volume V. Since all z-dependence disappears from the integral it is infinite at least due to the space-time integral over z giving the volume factor factor $(2T)V\to\infty$. One can try to make these diagrams finite by some regularisation method (such will be discussed later) but even better would be if we could get rid off them altogether. This is what we will argue below is possible.

A crucial insight is now that, in the denominator above, the figur-eight diagram will appear once at λ level, at λ^2 there will many diagrams but one of them will be just the product of two figur-eight diagrams, at pattern that continues at higher orders. However, n figur-eight diagrams making up one term in the diagram expansion has a symmetry factor $s=n!$ since all the figur-eight subdiagrams are identical. This implies that they will exponentiate to e^{V_∞}. In fact, this will happen even if all these diagrams contains a common other bubble diagram and hence e^{V_∞} factorises out of all these diagrams. This argument can be repeated until all kinds of vacuum bubble diagrams have been exponentiated and factorised. The end result is thus that the whole denominator becomes

$$\langle 0|e^{-i\int d^4z\mathcal{H}_I}|0\rangle = e^{\Sigma_{all\ bubbles}V_{bubble}}. \tag{1.515}$$

This result is rather amazing in itself but the punch line is instead that also in the numerator, for the same reason, these vacuum bubble diagrams exponentiate and factorise, and can hence be canceled against the ones in the denominator. Thus the vacuum bubbles are completely eliminated. The result is the **amazing formula no 2**:

$$\begin{aligned}\langle\Omega|T\phi^{exact}(x_n)...\phi^{exact}(x_1)|\Omega\rangle &= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\phi(x_n)...\phi(x_1)e^{-i\int d^4z\mathcal{H}_I}|0\rangle}{\langle 0|e^{-i\int d^4z\mathcal{H}_I}|0\rangle}\\ &= \lim_{T\to\infty(1-i\epsilon)}\langle 0|T\phi(x_n)...\phi(x_1)e^{-i\int d^4z\mathcal{H}_I}|0\rangle|_{\text{connected diagrams}},\end{aligned} \tag{1.516}$$

where *connected diagrams* refers to all diagrams without vacuum bubbles.

Let us now consider the 4-point function at order λ

$$\langle 0|T\phi(x_4)\phi(x_3)\phi(x_2)\phi(x_1)(\phi(z))^4|0\rangle \tag{1.517}$$

$$= (D_F(x_1-x_2)D_F(x_3-x_4)+...)(D_F(z-z))^2 \tag{1.518}$$

$$+ (D_F(x_1-z)D_F(x_2-z)D_F(x_3-x_4)+...)D_F(z-z) \tag{1.519}$$

$$+ D_F(x_1-z)D_F(x_2-z)D_F(x_3-z)D_F(x_4-z). \tag{1.520}$$

Here the first line contains the three terms representing the three ways to connect the four external point to pairwise each other via two propagators multiplied the figure-eight diagram while the second line consists of three terms all with one direct propagator and one snail-diagram. The last line, however, is the most important one since all external points $x_1, .., x_4$ are connected to each other via the interaction point z and thus there are no propagators going straight through the diagram. Such a diagram is called **fully connected** and is the only one in this case that describes non-trivial scattering. We will come back to these diagrams later when discussing scattering amplitudes.

Using the Feynman rule to write down for the fully connected four-point diagram we get (with the momenta p_1 and p_2 chosen as ingoing)

$$= (\frac{-i\lambda}{4!})\int d^4z 4! D_F(z-x_1)D_F(z-x_2)D_F(x_3-z)D_F(x_4-z)$$

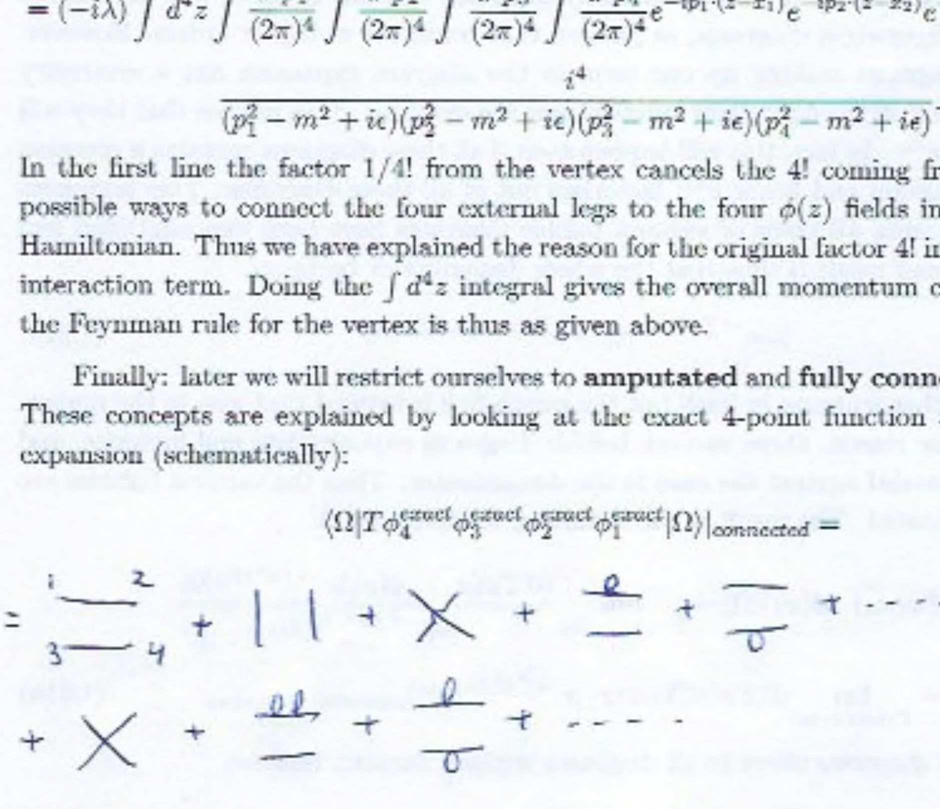
$$= (-i\lambda)\int d^4z \int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\int\frac{d^4p_4}{(2\pi)^4} e^{-ip_1\cdot(z-x_1)}e^{-ip_2\cdot(z-x_2)}e^{-ip_3\cdot(x_3-z)}e^{-ip_4\cdot(x_4-z)}$$

$$\frac{i^4}{(p_1^2-m^2+i\epsilon)(p_2^2-m^2+i\epsilon)(p_3^2-m^2+i\epsilon)(p_4^2-m^2+i\epsilon)}. \tag{1.521}$$

In the first line the factor $1/4!$ from the vertex cancels the $4!$ coming from counting all possible ways to connect the four external legs to the four $\phi(z)$ fields in the interaction Hamiltonian. Thus we have explained the reason for the original factor $4!$ in the Lagrangian interaction term. Doing the $\int d^4z$ integral gives the overall momentum conservation and the Feynman rule for the vertex is thus as given above.

Finally: later we will restrict ourselves to **amputated** and **fully connected** diagrams. These concepts are explained by looking at the exact 4-point function and its diagram expansion (schematically):

$$\langle\Omega|T\phi_4^{exact}\phi_3^{exact}\phi_2^{exact}\phi_1^{exact}|\Omega\rangle|_{connected} = \tag{1.522}$$



fully connected
and amputated