1.11.5 Cross sections and S-matrix elements

The purpose of this section is to obtain an expression for the quantity measured in collider experiments, the cross section σ, in terms of the matrix element of $U(T,-T)$ between the in state $|\mathbf{k}_1, \mathbf{k}_2\rangle$ and the out state $\langle \mathbf{p}_1, ..., \mathbf{p}_n|$ as $T \to \infty$. The connection to the n-point functions discussed previously, expressed via the amazing formulas

$$\langle\Omega|T\phi^{exact}(x_1)...\phi^{exact}(x_n)|\Omega\rangle = \frac{\langle 0|T\phi(x_1)...\phi(x_n)e^{-i\int d^4z\mathcal{H}_I}|0\rangle}{\langle 0|e^{-i\int d^4z\mathcal{H}_I}|0\rangle}$$

$$= \langle 0|T\phi(x_1)...\phi(x_n)e^{-i\int d^4z\mathcal{H}_I}|0\rangle|_{\text{connected diagrams}}, \tag{1.523}$$

can be stated as follows: For a matrix element corresponding to a pair of particles scattering into n particles, $2 \to n$, consider the $n+2$-point function and pick up the term that has the time-ordering of the ϕ fields with two of them in the infinite past and n of them in the infinite future. Then in a sense to be made more precise below we have

$$\langle 0|T\phi(x_1)...\phi(x_n)e^{-i\int d^4z\mathcal{H}_I}|0\rangle|_{\text{connected diagrams}} \to \langle \mathbf{p}_1, ..., \mathbf{p}_n|e^{-i\int d^4z\mathcal{H}_I}|\mathbf{k}_1, \mathbf{k}_2\rangle := S_{\text{out,in}}, \tag{1.524}$$

where we also made the relation to the S-matrix explicit. The details of this step is provided as part of the so called LSZ reduction procedure explained in PS section 7.2 which is not part of this course.

We now turn to a discussion and derivation of the quantity, the cross section, that is measured in a collider experiment as, e.g., at LHC at CERN. Consider a fixed target consisting of a bunch of particles of type $\mathcal{A}$ the number of them being $N_A = \rho_A \times l_A \times A$ where A is the area perpendicular to the line of motion of the particles in bunch $\mathcal{B}$ hitting those in the target. Also, ρ_A is the density in the bunch (which we assume constant through out the bunch) and l_A is the length of the bunch. The number of particles in bunch $\mathcal{B}$ is similarly $N_B = \rho_B \times l_B \times A$. The total number of scattering events N must be proportional to $N_A \times N_B$, a fact that suggests the following definition of the *cross section* σ (note that the area A is common to both bunches) and hence σ can roughly be thought of as the effective size of the target:

$$N = \frac{\sigma}{A}N_A N_B \Rightarrow \sigma = \frac{N}{\rho_A l_A \rho_B l_B A}. \tag{1.525}$$

To obtain a formula for σ we need to get an expression for the probability $\mathcal{P}(AB \to 12...n)$, that is, for the scattering of particles $\mathcal{A}$ and $\mathcal{B}$ to produce particles 1 to n. With such a formula one can then ask other more precise questions like the differential cross section for $\mathcal{A}$ and $\mathcal{B}$ to scatter into a specific set of particles with specified momenta $\mathbf{p_1}, ..., \mathbf{p_n}$:

$$d\mathcal{P}(AB \to 12...n) = (\Pi_{i=1}^n \frac{d^3p_i}{(2\pi)^3}\frac{1}{2E_i})|_{out}\langle \mathbf{p}_1..\mathbf{p}_n|\phi_A\phi_B\rangle_{in}|^2. \tag{1.526}$$

Here we express this differential probability in terms of the absolute square of the overlap of the incoming state $|\phi_A\phi_B\rangle$ and the outgoing one $\langle \mathbf{p}_1..\mathbf{p}_n|$. These states are quite different in nature: while the outgoing state is given by precise values of the measured momenta satisfying the standard QFT normalisation

$$\langle \mathbf{p}_1..\mathbf{p}_n|\mathbf{p'}_1..\mathbf{p'}_n\rangle = \Pi_{i=1}^n (2\pi)^3 2E_{\mathbf{p}_i}\delta^3(\mathbf{p}_i - \mathbf{p'}_i), \tag{1.527}$$

the definition of the incoming state reflects the non-zero width of the bunches $\mathcal{A}$ and $\mathcal{B}$. This is done by giving each bunch a distribution function $\phi(\mathbf{r})$ normalised to one, i.e., $\int d^3r|\phi(\mathbf{r})|^2 = 1$ implying also that $\int \frac{d^3k}{(2\pi)^3}|\phi(\mathbf{k})|^2 = 1$. The Fourier transform of $\phi(\mathbf{r})$ is denoted $\phi(\mathbf{k})$ and can be used to construct the incoming state using

$$|\phi\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\phi(\mathbf{k})|\mathbf{k}\rangle. \tag{1.528}$$

Note that although $|\mathbf{k}\rangle$ is an exact state in the interacting theory it is normalised as usual, that is in the same way as $|\mathbf{p}\rangle$ in the outgoing state given above. This then implies that $\langle\phi|\phi\rangle = 1$. Far in the past the bunches $\mathcal{A}$ and $\mathcal{B}$ are far apart also in space and we can assume that

$$|\phi_A\phi_B\rangle = |\phi_A\rangle\,|\phi_B\rangle, \tag{1.529}$$

where each factor state is of the form given above. This formula is valid only for zero impact parameter which we should not assume is the case. In fact, we must sum over all possible impact parameters obtained by shifting the $\mathcal{B}$ bunch by a vector $\mathbf{b}$ in the plane perpendicular to the line of motion of the $\mathcal{B}$ bunch. This means that while the $\mathcal{A}$ bunch state $|\phi_A\rangle$ is given by the expression for $|\phi\rangle$ above, the $\mathcal{B}$ bunch state now is

$$|\phi_B(\mathbf{b})\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}e^{-i\mathbf{b}\cdot\mathbf{k}}\phi_B(\mathbf{k})|\mathbf{k}\rangle, \tag{1.530}$$

where the factor $e^{-i\mathbf{b}\cdot\mathbf{k}}$ is the Fourier transform of the translation operator $e^{-\mathbf{b}\cdot\nabla}$ which moves the distribution function, and thus the bunch, $\phi_B(\mathbf{r})$ a distance $\mathbf{b}$. To get the complete probability we must integrate over $\mathbf{b}$ in the final expression.

Thus the number of scattering events is given by the number of $\mathcal{B}$ particles hitting each $\mathcal{A}$ particle (i.e., with $N_A = \rho_A l_A A = 1$) times the probability for a scattering to occur

$$dN = \int d^2b\, n_B\, d\mathcal{P}(\mathbf{b}) \tag{1.531}$$

where $n_B = \rho_B l_B$ is the particle density in the plane perpendicular to the incoming direction. Thus we get, assuming n_b is constant, for each $\mathcal{A}$ particle

$$d\sigma = \frac{dN}{\rho_B l_B \rho_A l_A A} = \frac{dN}{n_B} = \int d^2b\, d\mathcal{P}(\mathbf{b}). \tag{1.532}$$

This gives the final expression

$$d\sigma = (\Pi_{i=1}^n \frac{d^3p_i}{(2\pi)^3}\frac{1}{2E_i})\int d^2b \int \frac{d^3k_A}{(2\pi)^3}\frac{\phi(\mathbf{k}_A)}{\sqrt{2E_{\mathbf{k}_A}}}\int \frac{d^3\bar{k}_A}{(2\pi)^3}\frac{\phi^*(\bar{\mathbf{k}}_A)}{\sqrt{2E_{\bar{\mathbf{k}}_A}}}\int \frac{d^3k_B}{(2\pi)^3}\frac{\phi(\mathbf{k}_B)}{\sqrt{2E_{\mathbf{k}_B}}}\int \frac{d^3\bar{k}_B}{(2\pi)^3}\frac{\phi^*(\bar{\mathbf{k}}_B)}{\sqrt{2E_{\bar{\mathbf{k}}_B}}}$$

$$\times\, e^{i\mathbf{b}\cdot(\bar{\mathbf{k}}_B - \mathbf{k}_B)}({}_{out}\langle\mathbf{p}_1..\mathbf{p}_n|\mathbf{k}_A\mathbf{k}_B\rangle_{in})({}_{in}\langle\bar{\mathbf{k}}_A\bar{\mathbf{k}}_B|\mathbf{p}_1..\mathbf{p}_n\rangle_{out}) \tag{1.533}$$

where all the integrals involving $\bar{\mathbf{k}}$ should now be performed.

In order to do these integrals we extract the momentum conservation delta-functions from the overlaps and define the remaining quantity as the *matrix element* $\mathcal{M}$:

$$_{out}\langle \mathbf{p}_1..\mathbf{p}_n|\mathbf{k}_A\mathbf{k}_B\rangle_{in} = i\mathcal{M}(k_Ak_B \to p_1...p_n)(2\pi)^4\delta^4(\Sigma\, k - \Sigma\, p) \tag{1.534}$$

$$(_{out}\langle \mathbf{p}_1..\mathbf{p}_n|\bar{\mathbf{k}}_A\bar{\mathbf{k}}_B\rangle_{in})^* = -i\mathcal{M}^*(\bar{k}_A\bar{k}_B \to p_1...p_n)(2\pi)^4\delta^4(\Sigma\, \bar{k} - \Sigma\, p) \tag{1.535}$$

The name *matrix element* for the quantity $\mathcal{M}$ has its origin in the equation

$$_{out}\langle \mathbf{p}_1..\mathbf{p}_n|\mathbf{k}_A\mathbf{k}_B\rangle_{in} = {}_S\langle \mathbf{p}_1..\mathbf{p}_n, t_0|\mathbf{k}_A\mathbf{k}_B, t_0\rangle_S = {}_S\langle \mathbf{p}_1..\mathbf{p}_n, T|e^{-iH(2T)}|\mathbf{k}_A\mathbf{k}_B, -T\rangle_S|_{T\to\infty} \tag{1.536}$$

where we have gone from the Heisenberg picture to the Schrödinger picture, now defined at a common time due to the time translation operator. Taken a further step to the interaction picture it reads

$$_{out}\langle \mathbf{p}_1..\mathbf{p}_n|\mathbf{k}_A\mathbf{k}_B\rangle_{in} = {}_I\langle \mathbf{p}_1..\mathbf{p}_n, t=\infty|S|\mathbf{k}_A\mathbf{k}_B, t=-\infty\rangle_I := S_{fi}, \tag{1.537}$$

with the indices on the S-matrix refering to *initial* and *final.* Hence, writing $S = 1 + iT$ we can consider $\mathcal{M}$ to be the matrix elements, or scattering amplitude, of the T-matrix between states (with no suffixes *in* or *out*). This will be further discussed later.

The next step is to do the $\int d^2b$ integrals which gives $(2\pi)^2\delta^2(\bar{\mathbf{k}}_B^\perp - \mathbf{k}_B^\perp)$. Thus the integrals in the two $\bar{\mathbf{k}}_B^\perp$ directions are trivial putting $\bar{\mathbf{k}}_B^\perp = \mathbf{k}_B^\perp$ in $d\sigma$ above. As a second step we perform the corresponding $\bar{\mathbf{k}}_A^\perp$ integrals using two of the delta functions in the definition of $\mathcal{M}^*$ above. This will implement the relations

$$\bar{\mathbf{k}}_A^\perp = \Sigma_i \mathbf{p}_i^\perp - \bar{\mathbf{k}}_B^\perp = \Sigma_i \mathbf{p}_i^\perp - \mathbf{k}_B^\perp = \mathbf{k}_A^\perp, \tag{1.538}$$

where the last equality follows from using the information in the delta-functions in the definition of $\mathcal{M}$.

At this point there are two $\bar{k}$ integrals left to do: $\int d\bar{k}_A^z \int d\bar{k}_B^z$. Using the remaining two delta-functions from the $\mathcal{M}^*$ definition we get

$$\int d\bar{k}_A^z \int d\bar{k}_B^z \delta(\bar{E}_A + \bar{E}_B - \Sigma_i E_i)\delta(\bar{k}_A^z + \bar{k}_B^z - \Sigma_i p_i^z) = \tag{1.539}$$

$$\int d\bar{k}_A^z \delta(\bar{E}_A + \bar{E}_B - \Sigma_i E_i)|_{\bar{k}_B^z = \Sigma_i p_i^z - \bar{k}_A^z}. \tag{1.540}$$

To do the last integral we need to recall that $\bar{E}_A = \sqrt{\bar{\mathbf{k}}_A^2 + m_A^2}$ and $\bar{E}_B = \sqrt{\bar{\mathbf{k}}_B^2 + m_B^2}$ so that when viewing the argument of the delta function as a function of the integration variable $\bar{k}_A^z$ we get two contributions to the derivative of $\bar{E}_A + \bar{E}_B - \Sigma_i E_i$, namely $\frac{\partial \bar{E}_A}{\partial \bar{k}_A^z}$ and $\frac{\partial \bar{E}_B}{\partial \bar{k}_A^z} = -\frac{\partial \bar{E}_B}{\partial \bar{k}_B^z}$ where we used the fact that $\bar{k}_B^z = \Sigma_i p_i^z - \bar{k}_A^z$. We then directly find

$$\int d\bar{k}_A^z \delta(\bar{E}_A + \bar{E}_B - \Sigma_i E_i)|_{\bar{k}_B^z = \Sigma_i p_i^z - \bar{k}_A^z} = \frac{1}{|\frac{\bar{k}_A^z}{\bar{E}_A} - \frac{\bar{k}_B^z}{\bar{E}_B}|} := \frac{1}{|v_A - v_B|}. \tag{1.541}$$

Note that the last equality is a bit tricky since we only know that $\bar{\mathbf{k}}_A^\perp = \mathbf{k}_A^\perp$, $\bar{\mathbf{k}}_B^\perp = \mathbf{k}_B^\perp$ together with $\bar{E}_A + \bar{E}_B = \Sigma_i E_i = E_A + E_B$ and $\bar{k}_A^z + \bar{k}_B^z = \Sigma_i p_i^z = k_A^z + k_B^z$. We will however assume that this means that $\bar{\mathbf{k}}_A = \mathbf{k}_A$ and the same for B. This is certainly the most natural solution to these conditions on the momenta.

Using this information we can perform the $\int d^3k_A$ and $\int d^3k_B$ integrals provided we can argue that in the delta-functions occurring in the definition of $\mathcal{M}$ we can use the central values $p_A + p_B$, as defined by the momentum distribution $\phi(\mathbf{k})$, instead of $k_A + k_B$. This argument is given in the book by PS and is based on detector properties in actual experiments. With this input the integrals are just the normalization conditions on the distribution functions and the final answer becomes

$$d\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|} (\Pi_{i=1}^n \frac{d^3p_i}{(2\pi)^3} \frac{1}{2E_i}) \, |\mathcal{M}(p_A, p_B \to p_1, p_2 ... p_n)|^2 (2\pi)^4 \delta^4(p_A + p_B - \Sigma_i p_i). \tag{1.542}$$

Two important facts about this expression are:
1. The following parts of the $d\sigma$ is called the *n-body phase space*

$$\int \Pi_n = \int (\Pi_{i=1}^n \frac{d^3p_i}{(2\pi)^3} \frac{1}{2E_i}) (2\pi)^4 \delta^4(p_A + p_B - \Sigma_i p_i). \tag{1.543}$$

2. The whole expression for $d\sigma$ is Lorentz invariant except the prefactor

$$\frac{1}{2E_A 2E_B |v_A - v_B|} = \frac{1}{|E_B p_A^z - E_A p_B^z|} = \frac{1}{|\epsilon_{xy\mu\nu} p_A^\mu p_B^\nu|}, \tag{1.544}$$

which is boost invariant in the z direction and transforms as a cross sectional area in the xy directions. We can thus use this result in both the lab frame and the center of mass frame without any problems.

One special case is the $2 \to 2$ process which simplifies quite a bit, especially in the center of mass frame. In this case the incoming particles have total three-momentum equal to zero so the space delta-functions put also the sum of the outgoing momenta to zero. Thus we can trivially do the p_2 space integrals $\int d^3p_2$ to find

$$\int \Pi_2 = \int \frac{d^3p_1}{(2\pi)^3} \frac{1}{2E_1} \int \frac{d^3p_2}{(2\pi)^3} \frac{1}{2E_2} (2\pi)^4 \delta(E_A + E_B - E_1 - E_2) \delta^3(\mathbf{p}_1 + \mathbf{p}_2) =$$

$$\int \frac{dp_1 p_1^2 d\Omega}{(2\pi)^3 2E_1 2E_2} (2\pi) \delta(E_A + E_B - E_1 - E_2) = \int d\Omega \frac{p_1^2}{16\pi^2 E_1 E_2} (\frac{p_1}{E_1} + \frac{p_1}{E_2})^{-1} |_{p_1 = \tilde{p}_1} = \int d\Omega \frac{1}{16\pi^2} \frac{|\mathbf{p}_1|}{E_{cm}} |_{p_1 = \tilde{p}_1}, \tag{1.545}$$

where we used the fact that $E_{cm} = E_A + E_B = E_1 + E_2$ and where $\tilde{p}_1$ is the solution to $E_A + E_B = E_1 + E_2 = \sqrt{\mathbf{p}_1^2 + m_1^2} + \sqrt{\mathbf{p}_1^2 + m_2^2}$. Note that although the momenta cancel both for the incoming and outgoing pairs their energies are not the same since at this point all four masses are arbitrary. Inserting these results into the differential cross section formula gives

$$(\frac{d\sigma}{d\Omega})_{cm} = \frac{1}{2E_A 2E_B |v_A - v_B|} \frac{|\tilde{\mathbf{p}}_1|}{(2\pi)^2 4E_{cm}} |\mathcal{M}(p_A, p_B \to p_1, p_2)|^2. \tag{1.546}$$

Finally, by imposing that all four masses are the same this simplifies even further to

$$(\frac{d\sigma}{d\Omega})_{cm}(m_1 = m_2 = m_A = m_B = m) = \frac{|\mathcal{M}|^2}{64\pi^2 E_{cm}^2}, \tag{1.547}$$

where we have used that here $E_1 = E_2$ and $E_A = E_B$ and that the sum in both cases equals E_{cm}. We also used that $|v_A - v_B| = 2|v_A| = 2\frac{|\mathbf{p}_A|}{E_A} = 2\frac{|\mathbf{p}_{cm}|}{\frac{1}{2}E_{cm}} = 4\frac{|\mathbf{p}_{cm}|}{E_{cm}}$ and that all momenta $|\mathbf{p}|$ are equal and denoted $|\mathbf{p}_{cm}|$. This is the result quoted in Chapter 1 of PS.

Formally one can also apply the cross section formula to one "incoming" particle at rest to try to obtain a formula for the decay amplitude one this single particle. This procedure is not correct but does give the correct answer as is shown in PS Chapter 7, section 7.3 which is not part of the course. The formula obtained is

$$d\Gamma = \frac{1}{2m_A}(\Pi_{i=1}^n \frac{d^3 p_i}{(2\pi)^3}\frac{1}{2E_i})|\mathcal{M}(A \to p_1....p_n)|^2 (2\pi)^4 \delta^4(p_A - \Sigma_i p_i). \tag{1.548}$$

The final step in the process of computing cross sections is to relate the matrix element $\mathcal{M}$ defined by

$$_{out}\langle \mathbf{p}_1..\mathbf{p}_n|\mathbf{k}_A\mathbf{k}_B\rangle_{in} = i\mathcal{M}(k_A k_B \to p_1...p_n)(2\pi)^4\delta^4(\Sigma\, k - \Sigma\, p), \tag{1.549}$$

to the correlation functions that we know how to compute from before, namely

$$\langle 0|T\phi(x_1)....\phi(x_n)e^{-i\int d^4x\mathcal{H}_I}|0\rangle|_{connected}. \tag{1.550}$$

This will be done by in some sense pushing the the explicit ϕ fields either to the past or future infinity and there relate them to the *in* and *out* states. This leaves the operator $e^{-i\int d^4x\mathcal{H}_I}$ which will then correspond to the operator S whose matrix elements are denoted $\mathcal{M}$. The proof that this can be done rigorously is provided by the so called *LSZ reduction procedure*. This is the subject of PS section 7.2 which is not part of the course.

Briefly, however, this *LSZ* step can be described as follows. In a fully connected diagram (see end of the previous lecture) for a correlation function the $\phi(x)$ fields are contracted into a field at a vertex. The resulting propagator is then related to

$$\phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}e^{ip\cdot x}a^\dagger_{\mathbf{p}}|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{ip\cdot x}|\mathbf{p}\rangle_0. \tag{1.551}$$

Replacing $\phi(x)|0\rangle$ by $|\mathbf{p}\rangle_0$ will in the expression for the correlation function essentially correspond to dropping the integral over the external momentum together with the propagator and exponentials $e^{ip\cdot x}$. This means that the leg is amputated (see end of previous lecture) but such a leg can have any number of loops on it making it infinite when computed. The consequence of this fact is that an infinite constant, Z, will arise that is associated with renormalisation.

1.11.6 Computing S-matrix elements from Feynman diagrams

The formula for the differential cross section for $AB \to 12...n$ scattering was derived in the previous section and reads

$$d\sigma = \frac{1}{2E_A 2E_B|v_A - v_B|}(\Pi_{i=1}^n \frac{d^3p_i}{(2\pi)^3}\frac{1}{2E_i})\,|\mathcal{M}(p_A, p_B \to p_1, p_2...p_n)|^2(2\pi)^4\delta^4(p_A+p_B-\Sigma_i p_i). \tag{1.552}$$

The matrix element $\mathcal{M}$ that appears in this formula is defined in terms of the overlap between in-states and out-states by the following equation:

$$_{out}\langle \mathbf{p}_f|\mathbf{p}_i\rangle_{in} := i\mathcal{M}(\mathbf{p}_i \to \mathbf{p}_f)\,(2\pi)^4\delta^4(\Sigma p_f - \Sigma p_i), \tag{1.553}$$

where the two p^0s in the delta-functions are as usual given by the respective $E_{\mathbf{p}}$.

The important point here is that $_{out}\langle \mathbf{p}_f|\mathbf{p}_i\rangle_{in}$ and thus $\mathcal{M}$ can be related to the Feynman diagram expansion developed previously for correlation functions. The proof of this statement is known as the LSZ-reduction and is carried out in detail in Section 7.2 of PS. This section is not part of the course so instead we will look at some specific cases and this way try to understand how it works.

First we must be very precise about what the notation above means. The overlap $_{out}\langle \mathbf{p}_f|\mathbf{p}_i\rangle_{in}$ between the in-state and the out-state is written in the H-picture (the Heisenberg picture) so these states are time-independent. Instead, it is the operators that carry the time dependence. E.g., the operators relevant for us here are just quantised classical fields, like $\phi(t, \mathbf{r})$, which thus are in the H-picture. Also, in the H-picture the in- and out- states are time-independent eigenstates of the full Hamiltonian and are thus impossible to construct exactly in an interacting QFT.

Instead we will relate these states and the overlap $_{out}\langle \mathbf{p}_f|\mathbf{p}_i\rangle_{in}$ to the corresponding objects in the I-picture (the interaction picture). This way we will be able to see more explicitly what the S-matrix is. To do this we fix a time, say t_0, and declare that at that time the H-picture states are the same as the S-picture ones. Thus

$$_{out}\langle \mathbf{p}_f|\mathbf{p}_i\rangle_{in} := {}_S\langle t_0, \mathbf{p}_f|t_0, \mathbf{p}_i\rangle_S. \tag{1.554}$$

In the S-picture we can then evolve the ket-state backwards in time to $-T$ and the bra-state forward to $+T$ and then let $T \to \infty$. Using that time-evolution in the S-picture is given by $|t, \mathbf{p}\rangle_S := e^{-iH(t-t_0)}|t_0, \mathbf{p}\rangle_S$, we find

$$_{out}\langle \mathbf{p}_f|\mathbf{p}_i\rangle_{in} := \lim_{T\to\infty} {}_S\langle T, \mathbf{p}_f|e^{-iH(2T)}| - T, \mathbf{p}_i\rangle_S. \tag{1.555}$$

This way one can view the S-matrix as

$$S_{fi} := \lim_{T\to\infty} {}_S\langle T, \mathbf{p}_f| + T, \mathbf{p}_i\rangle_S, \tag{1.556}$$

i.e., as the overlap between the in-state translated forward in time to $+T$ and there checked how it compares to the out-state.

We will now go half-way to the S-picture and get the overlap in the I-picture. The key idea here is to leave some time-dependence, due to H_0, with the operators as we have done before, and let the states get a time-dependence from the interaction part of the Hamiltonian, i.e., H_I. This is achieved by defining the I-picture states by

$$|t, \mathbf{p}\rangle_S := e^{-iH_0(t-t_0)}|t_0, \mathbf{p}\rangle_I. \tag{1.557}$$

This means that in the I-picture the operators regains time-dependence from H_0 precisely as we did it in the beginning of this discussion where ϕ_I was defined. To see what this implies consider again

$$\begin{aligned} {}_{out}\langle \mathbf{p}_f|\mathbf{p}_i\rangle_{in} &= \lim_{T\to\infty} {}_S\langle T, \mathbf{p}_f|e^{-iH(T-t_0))}e^{iH(-T-t_0))}|-T, \mathbf{p}_i\rangle_S \\ &= \lim_{T\to\infty} {}_I\langle T, \mathbf{p}_f|e^{iH_0(T-t_0))}e^{-iH(T-t_0))}e^{iH(-T-t_0))}e^{-iH_0(-T-t_0))}|-T, \mathbf{p}_i\rangle_I \\ &= \lim_{T\to\infty} {}_I\langle T, \mathbf{p}_f|U(T, t_0)(U(-T, t_0))^\dagger|-T, \mathbf{p}_i\rangle_I \\ &= \lim_{T\to\infty} {}_I\langle T, \mathbf{p}_f|U(T, -T)|-T, \mathbf{p}_i\rangle_I. \end{aligned} \tag{1.558}$$

This is a nice result since we know from before that $U(t, t_0)$ is expressible in terms of the interaction picture field ϕ_I and thus under full control. The tricky part here is the states since they have a time-dependence that comes from H_I and therefore are very complicated. There are two simplifications taking place at this point:
1) Since the in and out states are asymptotic ones (i.e., coming in from very far and measured as outgoing states also very far away) they will factorise as:

$$|\mathbf{p}_1, ..., \mathbf{p}_n\rangle_I = |\mathbf{p}_1\rangle_I....|\mathbf{p}_n\rangle_I. \tag{1.559}$$

2) Each single particle state $|\mathbf{p}\rangle_I$ can then be replaced by a free state $|\mathbf{p}\rangle_0$ provided we compensate by an infinite (multiplicative) constant as a result of the amputation of external legs (below). This constant will be dealt with, and understood, later in the context of the renormalisation procedure.

Discarding non-interacting processes by defining the S-matrix as $S = 1 + iT$ and thereby restricting ourselves to only matrix elements of iT, we have finally

$${}_I\langle \mathbf{p}_f|iT|\mathbf{p}_i\rangle_I = \lim_{T\to\infty} {}_0\langle \mathbf{p}_f|T(e^{-i\int_{-T}^{T} H_I(t)dt})|\mathbf{p}_i\rangle_0|_{\text{(fully connected, amputated, renormalised)}}. \tag{1.560}$$

This is an extremely nice formula simply because one can now compute the RHS to any desired level in the perturbation expansion. Remember, however, that trying to sum it up leads often to an asymptotic series which starts to diverge after a certain number of terms. If this happens one needs mathematics that has only in the last few years become important in this context, like resurgence and transseries.

To get familiar with this formula and how it works we consider some simple cases in ϕ^4 theory. Consider the simplest 2 to 2 scattering: notation $AB \to 12$. Then
1) at order 0 in the coupling constant λ:

$${}_0\langle \mathbf{p}_1, \mathbf{p}_2|\mathbf{p}_A\mathbf{p}_B\rangle_0 = \sqrt{2E_A 2E_B 2E_1 2E_2}\langle 0|a_1 a_2 a_A^\dagger a_B^\dagger|0\rangle. \tag{1.561}$$

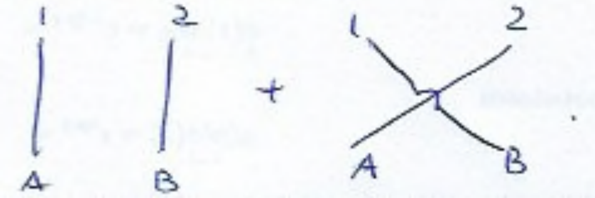
The vacuum to vacuum element can be computed using the usual commutator rules as follows

$$\langle 0|a_1 a_2 a_A^\dagger a_B^\dagger|0\rangle = \langle 0|[a_1 a_2, a_A^\dagger a_B^\dagger]|0\rangle = \langle 0|a_1[a_2, a_A^\dagger a_B^\dagger]|0\rangle + \langle 0|[a_1, a_A^\dagger a_B^\dagger]a_2|0\rangle$$

where the last term vanishes. Repeating this step this becomes

$$\begin{aligned}\langle 0|a_1 a_2 a_A^\dagger a_B^\dagger|0\rangle &= \langle 0|a_1 a_A^\dagger[a_2, a_B^\dagger]|0\rangle + \langle 0|a_1[a_2, a_A^\dagger]a_B^\dagger|0\rangle \\ &= (2\pi)^6(\delta^3(\mathbf{p}_1 - \mathbf{p}_A)\delta^3(\mathbf{p}_2 - \mathbf{p}_B) + \delta^3(\mathbf{p}_1 - \mathbf{p}_B)\delta^3(\mathbf{p}_2 - \mathbf{p}_A)).\end{aligned} \quad (1.562)$$

As before this result is nicely given in terms of diagrams as

$_0\langle \mathbf{p}_1 \mathbf{p}_2 | \mathbf{p}_A \mathbf{p}_B\rangle_0 =$ 

Note that we get only two diagrams in this case (there were three terms in the case of the 4-point correlation function)!

This term is thus part of the $\mathbf{1}$ in $S = \mathbf{1} + iT$ and is therefore discarded due to the *fully connected* requirement.

2) at first order in the coupling constant λ:

$${}_0\langle \mathbf{p}_1, \mathbf{p}_2|(-i\frac{\lambda}{4!})\int d^4z T(\phi^4(z))|\mathbf{p}_A\mathbf{p}_B\rangle_{0(\text{fully connected, amputated, renormalised})}. \quad (1.563)$$

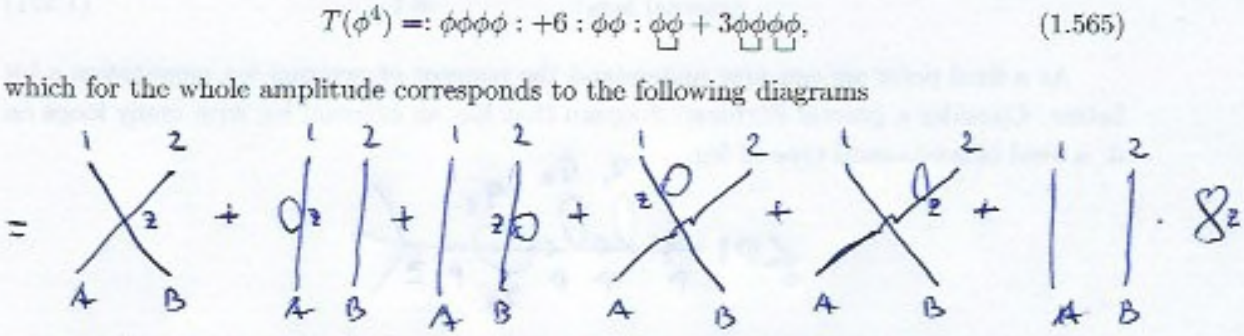
We thus need to analyse the expression

$${}_0\langle \mathbf{p}_1, \mathbf{p}_2|T\phi^4(z)|\mathbf{p}_A\mathbf{p}_B\rangle_0. \quad (1.564)$$

As for correlation functions one should first apply Wick's theorem but here all normal ordered terms do not vanish as they did when the states were vacuum states $|0\rangle$ and $\langle 0|$. Wick now tells us that (all fields at z)

$$T(\phi^4) =: \phi\phi\phi\phi : + 6 : \phi\phi : \underbrace{\phi\phi} + 3\underbrace{\phi\phi}\underbrace{\phi\phi}, \quad (1.565)$$

which for the whole amplitude corresponds to the following diagrams

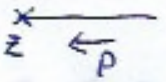


where only the first one is fully connected. Note that this diagram appears with a binomial factor $\binom{4}{2} = 6$ from $:\phi\phi\phi\phi: = ... + 6\phi^-\phi^-\phi^+\phi^+ + ...$ At the next stage we need $\phi^+(z)\phi^+(z)|\mathbf{p}_A\mathbf{p}_B\rangle_0$ and the analogues one for the out-states. We get two terms in each case since the A-state can contract with either of the two $\phi^+(z)$ giving a factor of two since they are identical. The answer is therefore for each contraction between a field from an interaction and an in-state

$$\phi^+(z)|\mathbf{p}_A\rangle_0 = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}a_\mathbf{p}e^{-ip\cdot z}\sqrt{2E_{\mathbf{p}_A}}a^\dagger_{\mathbf{p}_A}|0\rangle = e^{-ip_A\cdot z}|0\rangle := \phi(z)|\mathbf{p}_A\rangle_0. \quad (1.566)$$

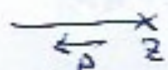
The last equality above represents a new kind of external leg Feynman rule. Taking this one step further we note that using this last equation for both in-states and out-states we can use $\langle 0|0\rangle = 1$ and therefore drop the vacuum states when stating the Feynman rules for external legs. Thus the rules are, for in-states

$$\phi(z)|\mathbf{p}\rangle_0 = e^{-ip\cdot z} = \quad (1.567)$$

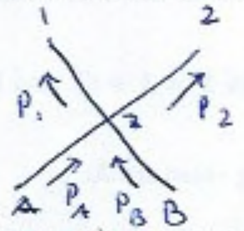


and for out-states

$$_0\langle\mathbf{p}|\phi(z) = e^{ip\cdot z} = \quad (1.568)$$



To finish the order λ^1 calculation above we draw the diagram with momenta and compute it as follows



$$= (4!)(-i\frac{\lambda}{4!})\int d^4z e^{-i(p_A+p_B-p_1-p_2)\cdot z} = -i\lambda(2\pi)^4\delta^4(p_A+p_B-p_1-p_2), \quad (1.569)$$

where the first factor 4! comes from the binomial coefficient $\binom{4}{2} = 6$ explained above multiplied by the two factors of 2 coming from $\phi^+(z)\phi^+(z)|\mathbf{p}_A\mathbf{p}_B\rangle_0$ and its out-state analogue.

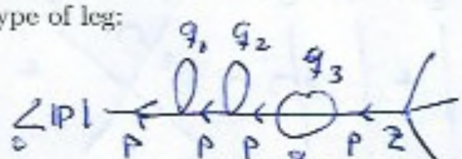
The Feynman rules for $i\mathcal{M}$ that give this final result immediately are

$$\text{vertex}: \quad i\mathcal{M} = -i\lambda, \quad (1.570)$$

and

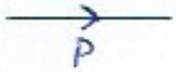
$$\text{external legs}: \qquad = 1. \quad (1.571)$$

As a final point we can now understand the concept of external leg *amputation* a bit better. Consider a general Feynman diagram that has an external leg with many loops on it, a kind of multi-snail type of leg:



In the scattering matrix context discussed here such a leg is contracted into a free in- or out-momentum state with its p^μ satisfying $E_\mathbf{p} = \sqrt{\mathbf{p}^2+m^2}$, that is $p^2 = m^2$. In the multi-snail external leg there are Feynman propagators between the loops that must therefore diverge since they are put on-shell by the external momentum state. It is therefore necessary to cut off this leg as close as possible to the interaction point at z and then later try to figure out if this is a valid procedure or not. We will not prove it in this course but the *field-renormalisation* to be discussed later will provide a positive, as well as plausible, answer to this question.

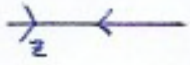
Let us summarise the final form of the Feynman rules for a real scalar field with ϕ^4 interactions:

1. For each propagator: 

$$= \frac{i}{p^2 - m^2 + i\epsilon}. \tag{1.572}$$

2. For each vertex: 

$$= -i\lambda. \tag{1.573}$$

3. For each external line: 

$$= 1. \tag{1.574}$$

4. Momentum conservation at each vertex. (1.575)

5. Integrate over undetermined momenta:
$$\int \frac{d^4p}{(2\pi)^4} \tag{1.576}$$

6. Divide by the symmetry factor s. (1.577)