1.11.7 Fermions and their Feynman rules

As we argued in the previous section by splitting the S-matrix as $S = 1 + iT$ and discarding non-interacting processes given by not fully connected diagrams we are restricting ourselves to matrix elements of iT:

$$ {}_I\langle \mathbf{p}_f | iT | \mathbf{p}_i \rangle_I = \lim_{T\to\infty} {}_0\langle \mathbf{p}_f | T(e^{-i\int_{-T}^{T} H_I(t)dt}) | \mathbf{p}_i \rangle_0 |_{\text{(fully connected, amputated, renormalised)}}, \tag{1.578} $$

where the RHS can be computed term by term in perturbation theory, i.e., in powers of the coupling constant appearing in H_I in the exponent. By identifying the RHS with

$$ i\mathcal{M}(p_i \to p_f)(2\pi)^4 \delta^4(\Sigma p_f - \Sigma p_i) \tag{1.579} $$

we obtain the matrix elements $i\mathcal{M}$ that we need in the computation of cross sections etc.

We are therefore interested in designing Feynman rules for the computation of fully connected and amputated diagrams. We found these rules for real scalar fields in the previous section and will here derive them for Dirac fermions and in the next section for the vector potential in electromagnetism.

Recall the mode expansions of Dirac fermions:

$$ \psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \Sigma_s \left(a^s_{\mathbf{p}} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}} v^s(p) e^{ip\cdot x} \right), \tag{1.580} $$

and

$$ \bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \Sigma_s \left(b^s_{\mathbf{p}} \bar{v}^s(p) e^{-ip\cdot x} + a^{s\dagger}_{\mathbf{p}} \bar{u}^s(p) e^{ip\cdot x} \right). \tag{1.581} $$

The Feynman propagator for the Dirac theory is, now with explicit spinor indices,

$$ \begin{aligned} (S_F(x_2 - x_1))_a{}^b &= \langle 0 | T \psi_a(x_2) \bar{\psi}^b(x_1) | 0 \rangle \\ &= \theta(x_2 - x_1) \langle 0 | \psi_a(x_2) \bar{\psi}^b(x_1) | 0 \rangle - \theta(x_1 - x_2) \langle 0 | \bar{\psi}^b(x_1) \psi_a(x_2) | 0 \rangle, \end{aligned} \tag{1.582} $$

which is also obtained from the contraction:

$$ (S_F(x_2 - x_1))_a{}^b = \underbrace{\psi_a(x_2)\bar{\psi}^b}(x_1) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p} + m)_a{}^b}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x_2 - x_1)}. \tag{1.583} $$

Recall also that the charge flow is from x_1 to x_2 since $\bar{\psi}(x_1)$ contains $a^\dagger$ and thus creates a particle at x_1. Also, $\underbrace{\psi(x_2)\psi}(x_1) = 0$ and $\underbrace{\bar{\psi}(x_2)\bar{\psi}}(x_1) = 0$.

To see how contractions work for anticommuting fields consider the following two examples (to simplify the notation we set $\psi(x_1) = \psi_1$ etc)

$$:\underbrace{\psi_1\psi_2\bar{\psi}_3}\bar{\psi}_4: = -\underbrace{\psi_1\bar{\psi}_3}:\psi_2\bar{\psi}_4: = -S_F(x_1 - x_3):\psi_2\bar{\psi}_4:, \tag{1.584} $$

and

$$:\underbrace{\psi_1\psi_2\bar{\psi}_3\bar{\psi}_4}: = +\underbrace{\psi_1\bar{\psi}_4}:\psi_2\bar{\psi}_3: = +S_F(x_1 - x_4):\psi_2\bar{\psi}_3:. \tag{1.585} $$

Before turning to a field theory example, namely Yukawa theory with scalars coupled to fermions, we need also to extract external leg contractions (the spacetime point z refers to the vertex here):

$$\psi(z)|\mathbf{p}, s\rangle_f := \psi^+(z)|\mathbf{p}, s\rangle_f = \int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}'}}}\Sigma_{s'}a^{s'}_{\mathbf{p}'}u^{s'}(p')e^{-ip'\cdot z}\sqrt{2E_{\mathbf{p}}}a^{s\dagger}_{\mathbf{p}}|0\rangle, \tag{1.586}$$

where we have used the split $\psi = \psi^+(a) + \psi^-(b^\dagger)$ with the operator content indicated. Here we use an index f on $|\mathbf{p}, s\rangle_f$ to show that the state is a one-fermion state, and not an anti-fermion state.

Turning the operator product $aa^\dagger|0\rangle$ into an anti-commutator and using the anti-commutation relations the above equation becomes

$$\psi(z)|\mathbf{p}, s\rangle_f := \psi^+(z)|\mathbf{p}, s\rangle_f = u^s(p)e^{-ip\cdot z}|0\rangle. \tag{1.587}$$

The same calculation for an anti-fermion, denoted $\bar{f}$, is in terms of operators b and $b^\dagger$ and thus uses $\bar{\psi} = \bar{\psi}^+(b) + \bar{\psi}^-(a^\dagger)$:

$$\bar{\psi}(z)|\mathbf{p}, s\rangle_{\bar{f}} := \bar{\psi}^+(z)|\mathbf{p}, s\rangle_{\bar{f}} = \bar{v}^s(p)e^{-ip\cdot z}|0\rangle. \tag{1.588}$$

Fermions carrying charge will follow a solid line with an arrow on it pointing in the direction of charge flow, i.e., from an in-coming fermion to an out-going fermion or from an out-going anti-fermion to an incoming anti-fermion. In an annihilation process the charged line would go from an in-coming fermion line to an out-going line for the in-coming anti-fermion and similarly for an pair-creation process.

Taking the Dirac conjugates of the states above we get

$${}_f\langle\mathbf{p}, s|\bar{\psi}(z) := {}_f\langle\mathbf{p}, s|\bar{\psi}^-(z) = \langle 0|\bar{u}^s(p)e^{ip\cdot z}. \tag{1.589}$$

The same calculation for an anti-fermion, denoted $\bar{f}$, gives

$${}_{\bar{f}}\langle\mathbf{p}, s|\psi(z) := {}_{\bar{f}}\langle\mathbf{p}, s|\psi^-(z) = \langle 0|v^s(p)e^{ip\cdot z}. \tag{1.590}$$

These states can now be combined to produce the following Feynman rules for entire fermion lines going through any diagram from the incoming ket-state via the interaction point at z to the out-going bra-state:

$${}_f\langle\mathbf{k}, r|\bar{\psi}(z)\psi(z)|\mathbf{p}, s\rangle_f = \langle 0|\bar{u}^r(k)e^{ik\cdot z}u^s(p)e^{-ip\cdot z}|0\rangle = \bar{u}^r(k)u^s(p)e^{-i(p-k)\cdot z}. \tag{1.591}$$

Thus we see that the vacuum states disappear. The rule is now that the factor $e^{-i(p-k)\cdot z}$ will be combined with the other similar factors related to the vertex at z which is integrated over and thus produces a momentum delta-function $\delta^4(\Sigma p_i)$ associated to the vertex. In drawing a Feynman diagram with a fermion line one thus puts a $u^s(p)$ where the fermion line starts and a $\bar{u}^r(k)$ where it ends. The situation for anti-fermions is similar but that the charge line goes in the opposite direction (so the ket-state is now the out-going fermion).

There is another issue popping up here: how are we to contract the spinors? In many cases it is clear how to do this but if there are many spinorial factors present and/or gamma matrices inserted between the spinors (e.g., Dirac propagators or from vertices in EM) one should write out the spinor indices $a, b, c, ...$ explicitly to keep track of matrix multiplications. Recall that

$$\psi(z)|\mathbf{p}, s\rangle_f := \psi^{(+)}(z)|\mathbf{p}, s\rangle_f = u^s(p)e^{-ip\cdot z}|0\rangle. \tag{1.592}$$

which therefore should be written

$$\psi_a(z)|\mathbf{p}, s\rangle_f := \psi_a^{(+)}(z)|\mathbf{p}, s\rangle_f = u_a^s(p)e^{-ip\cdot z}|0\rangle. \tag{1.593}$$

This means that γ^μ matrices will have indices $(\gamma^\mu)_a{}^b$ and Dirac conjugate spinors an upper index $\bar{\psi}^a$ (as already noted above).

For an anti-fermion line one gets (note the need for the spinor indices)

$${}_{\bar{f}}\langle\mathbf{p}, s|\psi_a(z)\bar{\psi}^b(z)|\mathbf{k}, r\rangle_{\bar{f}} = \langle 0|v_a^s(p)e^{ip\cdot z}\bar{v}^{rb}(k)e^{-ik\cdot z}|0\rangle = \bar{v}^{rb}(k)v_a^s(p)e^{-i(k-p)\cdot z}. \tag{1.594}$$

We will now use the **Yukawa theory** to set up a concrete calculation of a Feynman diagram representing the scattering of two fermions. The Lagrangian is

$$\mathcal{L}(\phi, \psi) = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_\phi^2\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m_\psi)\psi - g\phi\bar{\psi}\psi. \tag{1.595}$$

One direct application that springs to mind here is that if the scalar field develops a VEV (if we add a ϕ^4 term to get a mexican hat type potential for negative m_ϕ^2) then the fermion will become massive even if the original fermion mass term in the Lagrangian above is not present. This is what happens in the standard model since there the fermions are massless Weyl fermions prior to the Higgs effect taking place. Recall that the standard model in not symmetric between ψ_L and ψ_R.

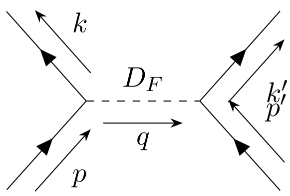
The interaction term in the Hamiltonian is (with all fields being in the interaction picture as usual)

$$\mathcal{H}_I(z) = g\phi(z)\bar{\psi}(z)\psi(z), \tag{1.596}$$

where the interaction point is at z^μ. This means that the two vertices in the scattering amplitude below come from the second order term in the expansion of

$$e^{-ig\int d^4z\phi(z)\bar{\psi}(z)\psi(z)}. \tag{1.597}$$

From this $\mathcal{H}_I$ we find the following 2 to 2 fermion scattering to lowest order



(1.598)

$$= \frac{1}{2}(-ig)^2 \int d^4z_1 \int d^4z_2 \langle \mathbf{k}, r(f); \mathbf{k}', r'(f)| T(\phi_1 \bar{\psi}_1 \psi_1 \phi_2 \bar{\psi}_2 \psi_2) |\mathbf{p}, s(f); \mathbf{p}', s'(f)\rangle. \quad (1.599)$$

Here we should note that the order of the fermions in each two-fermion external state is arbitrary but that there is a sign difference if the two one-particle states are flipped: $|\mathbf{p}, s(f); \mathbf{p}', s'(f)\rangle = -|\mathbf{p}', s'(f); \mathbf{p}, s(f)\rangle$ since the creation operators anticommute. A similar sign appears when hermitian conjugating a two-particle state $(|\mathbf{p}, s(f); \mathbf{p}', s'(f)\rangle)^\dagger = \langle \mathbf{p}, s(f); \mathbf{p}', s'(f)|$ but then the annihilation operators are in the opposite order to how the states appear in the notation of the bra-state (since $(a_1^\dagger a_2^\dagger |0\rangle)^\dagger = \langle 0|a_2 a_1)$. The important point is, however, that these signs do not matter if the same definitions of the two-particle states are used in all calculations, in particular when several diagrams are to be summed before the amplitude is squared to get the cross section.

Now let's compute the diagram above carefully. The first observation is that the external states do not contain any bosonic operators from ϕ so this part of the amplitude is just the scalar field Feynman propagator since

$$\langle 0|T(\phi_1 \phi_2)|0\rangle = D_F(z_1 - z_2), \quad (1.600)$$

where we have used Wick's theorem $T(\phi_1\phi_2) =: \phi_1\phi_2 : + \phi_1\phi_2$.

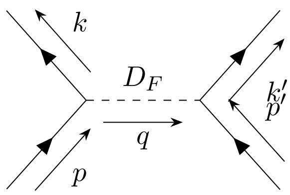
The more interesting part of the calculation is how the various fermionic fields from the vertices should be connected to the external states. This can be done in many ways, in fact! If we declare the left vertex to be vertex no 1 (i.e., at z_1) and the right vertex to be no 2 (i.e., at z_2) then the above diagram connects unprimed external one-particle states to z_1 and primed ones to z_2. This is expressed by the following contractions (recall $\langle 12| := \langle 0|a_2 a_1)$

$$\langle \mathbf{k}, r(f); \mathbf{k}', r'(f)| T(\bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2) |\mathbf{p}, s(f); \mathbf{p}', s'(f)\rangle = \quad (1.601)$$

$$= +(\langle \mathbf{k}', r'(f)|\bar{\psi}_2)(\langle \mathbf{k}, r(f)|\bar{\psi}_1)(\psi_1|\mathbf{p}, s(f)\rangle)(\psi_2|\mathbf{p}', s'(f)\rangle), \quad (1.602)$$

where moving around the anti-commuting objects gave rise to an even number of minus signs thus the final plus sign (recall the opposite order of fermions in the out-state).

Let us now complete the calculation of the above Feynman diagram. Using the external leg contractions derived above the diagram is



(1.603)

$$= \frac{1}{2}(-ig)^2 \int d^4z_1 \int d^4z_2 \Big(\bar{u}^r(k) u^s(p)\Big) \Big(\bar{u}^{r'}(k') u^{s'}(p')\Big) e^{-i(p-k)\cdot z_1} e^{-i(p'-k')\cdot z_2} D_F(z_2 - z_1).$$

$$(1.604)$$

Clearly each part of this Feynman diagram has its corresponding ingredient in this formula. We can, however, take it a couple of steps further by inserting the integral expression for the scalar field Feynman propagator

$$D_F(z_2 - z_1) = \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m_\phi^2 + i\epsilon} e^{-iq\cdot(z_2 - z_1)}. \tag{1.605}$$

Then the z_1 and z_2 integrals give two four-dimensional delta-functions, $(2\pi)^4\delta^4(p - k - q)(2\pi)^4\delta^4(p' - k' + q')$ which when integrated over q gives just $(2\pi)^4\delta^4(p - k + p' - k')$, i.e., the over-all momentum conservation condition. With these simplifications our diagram now reads

$$= \frac{1}{2}(-ig)^2 \Big(\bar{u}^r(k)u^s(p)\Big) \Big(\bar{u}^{r'}(k')u^{s'}(p')\Big) \Big(\frac{i}{q^2 - m_\phi^2 + i\epsilon}\Big)|_{q=p-k}(2\pi)^4\delta^4(p - k + p' - k'). \tag{1.606}$$

The final step is to equate this to $i\mathcal{M}(2\pi)^4\delta^4(p - k + p' - k')$ and read off the scattering amplitude $i\mathcal{M}$ (i.e., with the i). We find

$$i\mathcal{M} = -\frac{1}{2}ig^2 \frac{\Big(\bar{u}^r(k)u^s(p)\Big)\Big(\bar{u}^{r'}(k')u^{s'}(p')\Big)}{q^2 - m_\phi^2 + i\epsilon}|_{q=p-k}. \tag{1.607}$$

The Feynman rules that produce this expression directly will be given below.

We must now return to the contractions we performed to obtain the Feynman diagram we used so far. The point is that the expression coming from perturbation theory can be contracted in several other ways. First we could have flipped the two out-going one-particle states when contracting them into the two vertices. Doing this gives the following result

$$i\mathcal{M} = +\frac{1}{2}ig^2 \frac{\Big(\bar{u}^{r'}(k')u^s(p)\Big)\Big(\bar{u}^{r}(k)u^{s'}(p')\Big)}{q^2 - m_\phi^2 + i\epsilon}|_{q=p-k}, \tag{1.608}$$

where we note the important sign change of the whole result as well as the change in how the u spinors are contracted. Flipping instead the incoming two legs gives

$$i\mathcal{M} = +\frac{1}{2}ig^2 \frac{\Big(\bar{u}^r(k)u^{s'}(p')\Big)\Big(\bar{u}^{r'}(k')u^{s}(p)\Big)}{q^2 - m_\phi^2 + i\epsilon}|_{q=p-k}, \tag{1.609}$$

and flipping both the incoming and the outgoing pairs we find again the original overall minus sign

$$i\mathcal{M} = -\frac{1}{2}ig^2 \frac{\Big(\bar{u}^r(k)u^s(p)\Big)\Big(\bar{u}^{r'}(k')u^{s'}(p')\Big)}{q^2 - m_\phi^2 + i\epsilon}|_{q=p-k}. \tag{1.610}$$

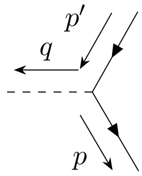
The results above are obviously pairwise the same which gives a factor 2 and only two different diagrams. Note that one way to understand why these diagrams are pairwise the same is to swap places of the two vertices. Then, e.g., in the last diagram one just undoes the two crossings of the legs and this diagram becomes identical to the first one analysed above. The same works for the two diagrams with one pair of crossing external legs. This factor of 2 will compensate the factor of $\frac{1}{2}$ associated with second order term in the expansion of the exponential.

The final result is thus

$$i\mathcal{M} = -ig^2 \left(\frac{\left(\bar{u}^r(k)u^s(p)\right)\left(\bar{u}^{r'}(k')u^{s'}(p')\right)}{q^2 - m_\phi^2 + i\epsilon}\Big|_{q=p-k} - \frac{\left(\bar{u}^r(k)u^{s'}(p')\right)\left(\bar{u}^{r'}(k')u^s(p)\right)}{q^2 - m_\phi^2 + i\epsilon}\Big|_{q=p-k} \right). \tag{1.611}$$

Of course, when computing the absolute value squared of this amplitude to get the cross section the sign outside the big bracket does not matter, what does matter is the relative sign between the two terms.

The Feynman rule for the vertex in the Yukawa theory was actually computed in the above the process:



$$= -ig \tag{1.612}$$

This corresponds to a fermion going from the in-state to the out-state.

However, to get this result we could also have considered the annihilation process $e^+e^- \to$*scalar boson*. This gives

$$= (-ig) \int d^4z_1 \langle \mathbf{k}|T(\phi_1\bar{\psi}_1\psi_1)|\mathbf{p}, s(f); \mathbf{p}', s'(\bar{f})\rangle. \tag{1.613}$$

Here we need the scalar contraction, with momentum leaving the closest vertex,

$$\langle \mathbf{k}|\phi(z) = \langle \mathbf{k}|\phi^{(-)}(z) = \langle 0|e^{ik\cdot z}. \tag{1.614}$$

Its conjugate is, with momentum entering the closest vertex

$$\phi(z)|\mathbf{k}\rangle = \phi^{(+)}(z)|\mathbf{k}\rangle = |0\rangle e^{-ik\cdot z}. \tag{1.615}$$

The annihilation vertex therefore becomes

$$= (-ig) \int d^4z_1 \langle 0|\bar{\psi}_1\psi_1)|\mathbf{p}, s(f); \mathbf{p}', s'(\bar{f})\rangle e^{ik\cdot z} \tag{1.616}$$

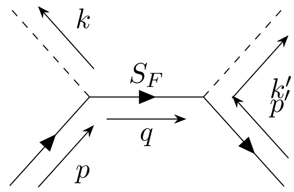
Here we should note that there are different ways to interpret this vertex. It's role in a bigger diagram can be one of the following four: since ψ contains a and $b^\dagger$ and $\bar{\psi}$ contains b and $a^\dagger$, the combination we are looking at here $\bar{\psi}\psi$ give non-zero results for any of the four choices of external states 1)the above case $\langle 0|\bar{\psi}\psi|f,\bar{f}\rangle$ using a and b, 2) $\langle f\bar{f}|\bar{\psi}\psi|0\rangle$, 3) $\langle f|\bar{\psi}\psi|f\rangle$, and $\langle \bar{f}|\bar{\psi}\psi|\bar{f}\rangle$.

The above process without scalar fields on the external legs provides an understanding for how the scalar propagator and its Feynman rule. Being quadratic in momenta this propagator does not have a direction and hence no arrow on it, and the momentum direction can be chosen arbitrarily. For complex fields there is a direction of charge flow indicated by an arrow on the propagator line from x_1 to x_2 corresponding to $D_F = \langle 0|\Phi(x_2)\bar{\Phi}(x_1)|0\rangle$.

This situation is a bit more intricate for fermions since they are linear in momenta so one should really check how charge and momenta directions arise by looking at a specific calculation. We therefore consider again the Yukawa model and compute the annihilation of e^+e^- into two scalar particles From the second order term in the expansion):

$$\frac{1}{2}(-ig)^2\int d^4z_1\int d^4z_2\langle k,k'|T(\phi_1\bar{\psi}_1\psi_1\phi_2\bar{\psi}_2\psi_2)|\mathbf{p},s(f);\mathbf{p}',s'(\bar{f})\rangle. \tag{1.617}$$

In this case we must contract two of the fermions between the two vertices as follows if we want to get the diagram



(1.618)

The contractions needed here are done on (moving first the bosonic fields)

$$\langle k,k'|T(\phi_1\bar{\psi}_1\psi_1\phi_2\bar{\psi}_2\psi_2)|\mathbf{p},s(f);\mathbf{p}',s'(\bar{f})\rangle = \langle k,k'|\phi_1\phi_2\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2)|\mathbf{p},s(f);\mathbf{p}',s'(\bar{f})\rangle, \tag{1.619}$$

so that by rearranging the fermionic field operators we get

$$\langle k,k'|\phi_1\phi_2\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2|\mathbf{p},s(f);\mathbf{p}',s'(\bar{f})\rangle = +\langle k,k'|\phi_1\phi_2\psi_2\bar{\psi}_1\bar{\psi}_2\psi_1|\mathbf{p},s(f);\mathbf{p}',s'(\bar{f})\rangle. \tag{1.620}$$

Note that the fermion propagator goes from vertex 1 to vertex 2 automatically once the external states have been contracted as indicated by the diagram. The charged fermion line therefore runs from the left incoming fermion leg through the propagator to the right anti-fermion incoming leg (where the arrow runs backwards).

What we want to find out now is how the momentum through the propagator is directed. So completing the calculation we get

$$\frac{1}{2}(-ig)^2\int d^4z_1\int d^4z_2\int\frac{d^4q}{(2\pi)^4}e^{-i(p-k)\cdot z_1}e^{-i(p'-k')\cdot z_2}\bar{v}^{s'}(p')\frac{i(\gamma^\mu q_\mu+m)}{q^2-m^2+i\epsilon}u^s(p)e^{-iq(z_2-z_1)}. \tag{1.621}$$

Doing the z_1 and z_2 integrals gives $(2\pi)^4\delta^4(p-k-q)$ and $(2\pi)^4\delta^4(p'-k'+q)$, respectively. Then we can use these delta-functions to do the q integrals which gives

$$\frac{1}{2}(-ig)^2\bar{v}^{s'}(p')\frac{i(\gamma^\mu q_\mu+m)}{q^2-m^2+i\epsilon}u^s(p)|_{q=p-k}(2\pi)^4\delta^4(p+p'-(k+k')). \tag{1.622}$$

So the convention for the Dirac propagator $S_F(z_2 - z_1)$ is that it runs from z_1 to z_2 with an arrow in this direction representing both charge and momentum. Thus with the matrix indices written out we have $\bar{u}^a(S_F(z_2 - z_1))_a{}^b u_b$ where one or both the us can be vs, the important thing being that the one with bar is to the left of S_F and the one without the bar to the right. Note that $\bar{u}^a(S_F(z_2 - z_1))_a{}^b u_b$ the charged line runs from past to future while $\bar{v}^a(S_F(z_2 - z_1))_a{}^b u_b$ is an annihilation process as in the example above. The last two possibilities have similar interpretations.

Also in this case one has to add two diagrams to get the complete amplitude at this order in the coupling constant: the one above and the one with one twisted outgoing pair of scalar legs. Note that the two diagrams add in this case.

Exercise: Give the contractions in $\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2|\mathbf{p}, s(f); \mathbf{p}', s'(\bar{f})\rangle$ which gives the diagram with the two incoming fermion legs crossed and show that this diagram has a plus sign relative the uncrossed diagram. This shows that the two vertices are equivalent and can be interchanged freely without sign flips. Any diagram with two vertices therefore gets an overall factor two from the interchanges of the two vertices. This property generalises to the nth term in the perturbative expansion (= the Taylor series expansion of the exponential) where the $\frac{1}{n!}$ factor is canceled by the $n!$ possible orderings of the vertices.

If the process is the opposite to the one above, i.e., two scalar particles annihilate and produce an e^+e^- pair as out-going particles, then we need to compute

$$\frac{1}{2}(-ig)^2 \int d^4z_1 \int d^4z_2 \langle \mathbf{p}, s(f); \mathbf{p}', s'(\bar{f})|T(\phi_1\bar{\psi}_1\psi_1\phi_2\bar{\psi}_2\psi_2)|k, k'\rangle, \tag{1.623}$$

where the fermionic contractions needed to obtain this process are (unprimed momenta at vertex z_1 and primed ones at z_2 with the charged line enters via z_2 and exits from z_1)

$$\langle \mathbf{p}, s(f); \mathbf{p}', s'(\bar{f})|\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2\phi_1\phi_2|k, k'\rangle, \tag{1.624}$$

which shows that the Dirac propagator now runs in the opposite direction to the case above, that is in this new case it runs from z_2 to z_1. Thus it is written $S_F(z_1 - z_2)$ and the arrow on its line again represents charge and momenta running in the same direction.
Exercise: Verify the last statement about the Dirac propagator.

To get the cross section for these processes requires computing the absolute square of the amplitudes above. For this we need some more gamma-matrix technology that we will explain in the next chapter in the context of QED.

There is one more very interesting aspect related to the signs obtained when moving the fermions around as we had to do in the computations above. This arises for instance if we consider the scattering of two scalar particles into two scalar particles at order g^4, i.e.,

$$\frac{1}{4!}(-ig)^4 \int d^4z_1 ... \int d^4z_4 \langle \mathbf{k}, \mathbf{k}'|T(\phi_1\bar{\psi}_1\psi_1.....\phi_4\bar{\psi}_4\psi_4|\mathbf{p}, \mathbf{p}'\rangle. \tag{1.625}$$

Since the scalar fields $\phi_1 ... \phi_4$ contract onto the four external scalar states the four fermionic fields must contract with each other. The fully connected diagram in this case is the loop diagram that arises if the fermions are contracted as follows

$$\langle 0|\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2\bar{\psi}_3\psi_3\bar{\psi}_4\psi_4|0\rangle. \tag{1.626}$$

To get four Dirac propagators from this expression we must move the first fermion $\bar{\psi}_1$ to the very end which gives a minus sign and thus it becomes

$$= -tr(S_F(z_3 - z_4)S_F(z_2 - z_3)S_F(z_1 - z_2)S_F(z_4 - z_1)), \tag{1.627}$$

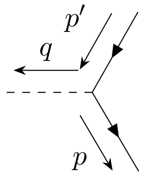
where $tr(...)$ refers to the trace over the spinor indices (remember that S_F is really a matrix $(S_F(z_2 - z_1))_a{}^b := \psi_a(z_2)\bar{\psi}^b(z_1))$. One can prove that a closed fermion loop always gives an extra minus sign compared to a bosonic loop.

Comment: SUSY. This minus sign from fermion loops is one of the basic facts that makes supersymmetry so interesting. Constructing carefully a theory consisting of e.g. a complex scalar field and a Majorana fermion one can get the bosonic loops to exactly cancel the fermionic loops and thus find a theory with much fewer infinities than in similar theories without supersymmetry.

Comment: see PS p. 121. The scattering of two fermions above via the exchange of a scalar particle can be compared to the Born approximation and seen to give an attractive *Yukawa potential*. Similarly the scattering of a fermion and an antifermion gives the same result. Thus the force mediated by the scalar field (spin 0) is always attractive, which can be compared to gravity (spin 2) which is also always attractive. In EM (spin 1 force field) this is no longer true as we will see later.

We can now sum up the fermionic Feynman rules in Yukawa theory:

1. For each propagator: $$= \frac{i(\not{p} + m)_a{}^b}{p^2 - m^2 + i\epsilon}. \tag{1.628}$$

2. For each vertex: 

$$= -ig\delta_a^b. \tag{1.629}$$

3. For external lines:
$$u_a^s(p), \qquad v_a^s(p), \tag{1.630}$$

$$\bar{u}^{sa}(p), \qquad \bar{v}^{sa}(p). \tag{1.631}$$

4. Momentum conservation at each vertex. (1.632)

5. Integrate over undetermined momenta: $$\int \frac{d^4p}{(2\pi)^4} \tag{1.633}$$

6. Divide by the symmetry factor s and determine the sign of each diagram. (1.634)

Since the scalar fields $\phi_1 ... \phi_4$ contract onto the four external scalar states the four fermionic fields must contract with each other. The fully connected diagram in this case is the loop diagram that arises if the fermions are contracted as follows

$$\langle 0|\bar\psi_1\psi_1\bar\psi_2\psi_2\bar\psi_3\psi_3\bar\psi_4\psi_4|0\rangle. \tag{1.626}$$

To get four Dirac propagators from this expression we must move the first fermion $\bar\psi_1$ to the very end which gives a minus sign and thus it becomes

$$= -tr(S_F(z_3 - z_4)S_F(z_2 - z_3)S_F(z_1 - z_2)S_F(z_4 - z_1)), \tag{1.627}$$

where $tr(...)$ refers to the trace over the spinor indices (remember that S_F is really a matrix $(S_F(z_2 - z_1))_a{}^b := \psi_a(z_2)\bar\psi^b(z_1)$). One can prove that a closed fermion loop always gives an extra minus sign compared to a bosonic loop.

Comment: SUSY. This minus sign from fermion loops is one of the basic facts that makes supersymmetry so interesting. Constructing carefully a theory consisting of e.g. a complex scalar field and a Majorana fermion one can get the bosonic loops to exactly cancel the fermionic loops and thus find a theory with much fewer infinities than in similar theories without supersymmetry.

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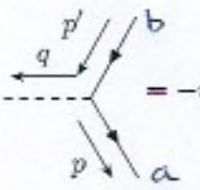
We can now sum up the fermionic Feynman rules in Yukawa theory:

1. For each propagator: $\underbrace{\psi_a \bar\psi^b} = a \longleftarrow b = \dfrac{i(\not p + m)_a{}^b}{p^2 - m^2 + i\epsilon}$. (1.628)

2. For each vertex:



$= -ig\delta_a^b$. (1.629)

3. For external lines:

$\underbrace{\psi_a|\mathbf{p},s\rangle}_f = \;\longleftarrow\; = u_a^s(p)$, $\quad {}_{\bar f}\langle \underbrace{\mathbf{p},s|\psi_a} = \;\longrightarrow\; = v_a^s(p)$. (1.630)

${}_f\langle \mathbf{p},s\underbrace{|\bar\psi^a} = \;\longleftarrow\; = \bar u^{sa}(p)$, $\quad \underbrace{\bar\psi^a|\mathbf{p},s\rangle}_{\bar f} = \;\longrightarrow\; = \bar v^{sa}(p)$. (1.631)

4. Momentum conservation at each vertex. (1.632)

5. Integrate over undetermined momenta: $\displaystyle\int \frac{d^4p}{(2\pi)^4}$ (1.633)

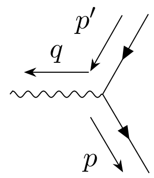
6. Divide by the symmetry factor s and determine the sign of each diagram. (1.634)

1.11.8 Feynman rules in QED and other field theories

We have finally come to a point where we can start analysing the field theory of QED. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi. \tag{1.635}$$

The interaction term is $\mathcal{L}_{int} = -eA_\mu\bar{\psi}\gamma^\mu\psi$ and hence $\mathcal{H}_I = eA_\mu\bar{\psi}\gamma^\mu\psi$ where the fields are the ones in the interaction picture although we don't use that notation. Similar to the Yukawa theory the vertex in QED is



$$= -ie(\gamma^\mu)_a{}^b. \tag{1.636}$$

The operators creating and destroying photon states are denoted $a^{\dagger\lambda}_{\mathbf{p}}$ and $a^\lambda_{\mathbf{p}}$. Here λ is running over the possible polarisations, i.e., possible independent degrees of freedom of the vector potential, which requires a separate analysis. This the mode expansion is

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\Sigma_\lambda(\epsilon^\lambda_\mu(p)a^\lambda_{\mathbf{p}}e^{-ip\cdot x} + \epsilon^{\star\lambda}_\mu(p)a^{\dagger\lambda}_{\mathbf{p}}e^{ip\cdot x}). \tag{1.637}$$

Note that since A_μ is a real field there are no b-operators.

The ϵ^λ is a (complex) polarisation tensor which we can specify by solving the maxwell equation in the Lorentz gauge. Thus imposing $\partial_\mu A^\mu = 0$, or in momentum space $p^\mu A_\mu = 0$, we can use this condition to express one component of A_μ in terms of the other three. The remaining gauge transformations (i.e., those not eliminated by the Lorentz condition are solution to $\Box\Lambda = 0$ where $\delta A_\mu = \partial_\mu\Lambda$)can the be used to eliminate one more of the A_μ components (since also $\Box A_\mu = 0$ in this gauge). Therefore the polarisation tensor has only two values for λ. We will leave this subtle point here and return to it when it becomes crucial to understand it better in the next chapter.

By comparison to the scalar propagator we can directly write down the photon propagator

$$= -\frac{ig_{\mu\nu}}{q^2 + i\epsilon}. \tag{1.638}$$

The reason the sign is opposite to the one for the scalar propagator is that the physical degrees of freedom in the vector potential really reside in its space components and $g_{ij} = -\delta_{ij}$. Again this will make more sense when we return to it in the next chapter. Finally, external photon lines come from the contractions

$$\overbracket{A_\mu|\mathbf{p}},\epsilon^\lambda_\mu\rangle = \epsilon^\lambda_\mu(p), \tag{1.639}$$

and

$$\langle\overbracket{\mathbf{p},\epsilon^\lambda_\mu|A}_\mu = \epsilon^{\star\lambda}_\mu(p). \tag{1.640}$$

We end by analysing Maxwell's equations and the role of gauge invariance. Maxwell's equation read

$$\partial_\mu F^{\mu\nu} = -j^\nu. \tag{1.641}$$

We have seen in the case of the other fields that p^2 and even $\gamma^\mu p_\mu$ can be used to derive Feynman propagators. This fails for the Maxwell equations since the operator in $\partial_\mu F^{\mu\nu} = \Box A^\nu - \partial^\nu(\partial_\mu A^\mu)$ now has the momentum expression $p^2\delta^\nu_\mu - p^\nu p_\mu$ which is not invertible since it has a zero eigenvalue solution p^μ. The way out of this problem is to use gauge invariance to impose the Lorentz condition $\partial_\mu A^\mu$ which turns the Maxwell operator into just $\Box$ which is invertible and the same as for the scalar field.

In momentum space this condition means $p^\mu \epsilon^\lambda_\mu = 0$. Also, the free Maxwell equations imply $p^2 = 0$. For a photon moving in the positive z-direction $p^\mu = E(1,0,0,1)$ and the Lorentz condition gives $\epsilon^\lambda_0 + \epsilon^\lambda_3 = 0$. The remaining gauge transformations reads now $\delta A_\mu = ip_\mu\Lambda$ which makes it possible to set $A_0 = 0$ and thus $A_3 = 0$. The photon therefore has only two physical degrees of freedom which in the case here lie in the xy-plane, or are right- or left-handed circular polarised: $\epsilon^\lambda_\mu = (0, 1, \pm i, 0)$. (Recall the same discussion in Chapter 1 in PS.)

The problem with vector potentials when quantised seems to come from the time component A_0 since it leads to negative normed states in the Hilbert space. If this happens probabililty is not conserved and quantum mechanics makes no sense. This is the real reason the gauge invariance is so important: it makes it possible to gauge away A_0 and thus solve the Hilbert space problem. As we will see in the next chapter this implies of course a new problem if we want our formalism, that is Feynman rules etc, to be Lorentz covariant. This rather deep problem is solved in the next chapter by introducing the so called **Ward identities**.

After this massive chapter we are now ready to apply all the rules derived here to physical problems and compute cross sections from first principles. Once the Feynman rules are understood and under control applying them is in fact not that difficult.