1.12 Computation of cross sections for some elementary processes

In this chapter we will start computing cross sections for scattering processes in QED. There are several interesting cases which can be divided into three groups

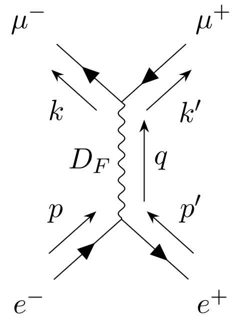
1) $e^+e^- \to e^+e^-$ called Bhabha scattering, or $\to \mu^+\mu^-$ (considered already), or $\to \tau^+\tau^-$ or $\to q\bar{q}$, where the first three cases involve leptons/anti-leptons from each of the three families (generations) and the last one any quark/anti-quark pair from any family.
Related processes (by crossing symmetry as explained later)
$e^-e^- \to e^-e^-$: Møller scattering, similarly for the other cases above.

2) Compton scattering: $e^-\gamma \to e^-\gamma$ given by the famous Klein-Nishina formula.
Related processes (by crossing symmetry as explained later)
Pair annihilation $e^+e^- \to 2\gamma$.

3) Scattering against fixed target:
a) Mott scattering $e^-\times \to e^-\times$ ($\times$=fixed target)
b) Annihilation and pair creation in the field of a fixed target
c) Bremsstrahlung $e^-\times \to e^-\gamma\times$ ($\times$=fixed target)

1.12.1 Computation of unpolarised scattering processes: $e^+e^- \to \mu^+\mu^-$

The first, and in some sense the simplest, process we can consider is the one discussed in the beginning of the course $e^+e^- \to \mu^+\mu^-$. The Feynman diagram is (the only one, compare to ϕ^4)



(1.642)

which directly gives the following expression by using the QED Feynman rules from the previous chapter

$$= \bar{u}^r(k)(-ie\gamma^\mu)v^{r'}(k')\left(\frac{-ig_{\mu\nu}}{q^2+i\epsilon}\right)\bar{v}^{s'}(p')(-ie\gamma^\nu)u^s(p). \tag{1.643}$$

Note that the spinors u and v in the first factor are the same as the ones in the last factor except for masses $k^2 = m_\mu^2$ and $p^2 = m_e^2$, and the same for the primed momenta.
Comment: This diagram could be written down in a Lorentz invariant form directly from the Feynman rules which is of course very nice. Thus we did not need to involve vacuum

states or operators of any kind! Can these rules perhaps be derived without going through the rather involved steps of non-Lorentz covariant second quantised perturbation theory (i.e., Chapter 4 in PS)? The answer is Yes! This is exactly what the Feynman path integral formulation provides. If you are interested, see Chap 9 in PS, but this chapter is not part of the course.

Simplifying the above expression a bit and equating it to the matrix element $i\mathcal{M}$ (note the i) we get

$$i\mathcal{M}(e^+e^- \to \mu^+\mu^-) = \frac{ie^2}{q^2+i\epsilon}\Big(\bar{v}^{s'}(p')\gamma^\mu u^s(p)\Big)\Big(\bar{u}^r(k)\gamma_\mu v^{r'}(k')\Big)|_{q=p+p'}. \tag{1.644}$$

To get the cross section we need to compute the absolute square of this matrix element, i.e., $|\mathcal{M}|^2 = \mathcal{M}^\star\mathcal{M}$. This requires getting the complex conjugates of the expressions in the brackets above:

$$\Big(\bar{v}^{s'}(p')\gamma^\mu u^s(p)\Big)^\dagger = \Big((v^{s'}(p'))^\dagger\gamma^0\gamma^\mu u^s(p)\Big)^\dagger = u^\dagger\gamma^{\mu\dagger}\gamma^0 v = \bar{u}\gamma^\mu v, \tag{1.645}$$

where we have used the familiar identity $\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu$. This step gives rise to the following rather nice expression

$$|\mathcal{M}|^2_{pol} = \frac{e^4}{(q^2+i\epsilon)^2}\Big(\bar{v}^{r'}(k')\gamma_\mu u^r(k)\Big)\Big(\bar{u}^r(k)\gamma_\nu v^{r'}(k')\Big)\Big(\bar{u}^s(p)\gamma^\mu v^{s'}(p')\Big)\Big(\bar{v}^{s'}(p')\gamma^\nu u^s(p)\Big)|_{q=p+p'}. \tag{1.646}$$

This expression is valid for any polarisations of the in-coming and out-going particles, thus the *pol* on $|\mathcal{M}|^2_{pol}$. It can be computed rather easily with a trick that we will come back to later. At this point it is better to do the unpolarised case first to see how one should handle expressions like this. As explained in the beginning of the course this means summing over the out-going polarisations and averaging over the in-coming ones as follows

$$|\mathcal{M}|^2_{unpol} = (\frac{1}{2}\Sigma_s\frac{1}{2}\Sigma_{s'})\Sigma_r\Sigma_{r'}|\mathcal{M}(e^+e^- \to \mu^+\mu^-)|^2_{pol} = \tag{1.647}$$

$$\frac{e^4}{4(q^2+i\epsilon)^2}\Sigma_{r,r'}\Big(\bar{v}^{r'}(k')\gamma_\mu u^r(k)\Big)\Big(\bar{u}^r(k)\gamma_\nu v^{r'}(k')\Big)\Sigma_{s,s'}\Big(\bar{u}^s(p)\gamma^\mu v^{s'}(p')\Big)\Big(\bar{v}^{s'}(p')\gamma^\nu u^s(p)\Big)|_{q=p+p'}. \tag{1.648}$$

It is here that the completeness relation for the spinors u and v that we derives in the Dirac chapter become extremely handy: first use $\Sigma_r u^r(k)\bar{u}^r(k) = \gamma^\rho k_\rho + m$ to get

$$\Sigma_{r,r'}\Big(\bar{v}^{r'}(k')\gamma_\mu u^r(k)\Big)\Big(\bar{u}^r(k)\gamma_\nu v^{r'}(k')\Big) = \Sigma_{r'}\Big(\bar{v}^{r'}(k')\gamma_\mu(\gamma^\rho k_\rho + m)\gamma_\nu v^{r'}(k')\Big). \tag{1.649}$$

Then the second step is the key: we can move $\bar{v}^{r'}(k')$ to the end of the expression and there use $\Sigma_{r'}v^{r'}(k')\bar{v}^{r'}(k') = \gamma^\rho k'_\rho - m$ to get a trace over the spinorial indices. That this is possible becomes clear by writing out the spinor indices and remembering that the v spinor is just a classical object and can be moved around freely. Thus the above expression simplifies to

$$tr\Big(\gamma_\mu(\gamma^\rho k_\rho + m)\gamma_\nu(\gamma^\sigma k'_\sigma - m)\Big). \tag{1.650}$$

Repeating these steps for the other factor (i.e., the other Dirac line in the Feynman diagram) we get (using $\not p := \gamma^\mu p_\mu$)

$$|\mathcal{M}|^2_{unpol} = \frac{e^4}{4(q^2+i\epsilon)^2} tr\Big(\gamma_\mu(\not k + m_{(\mu)})\gamma_\nu(\not k' - m_{(\mu)})\Big) tr\Big(\gamma^\mu(\not p' - m_{(e)})\gamma^\nu(\not p + m_{(e)})\Big), \quad (1.651)$$

where the indices on the masses $m_{(e)}$ and $m_{(\mu)}$ indicate which fermion line the trace belongs to.

To evaluate trace expressions like the ones above we need some traceology. Since we already know how to expand the product of any number of gamma matrices in the gamma basis the problem of computing any trace is reduced to verifying that it is only the unit matrix that has a non-zero trace. The gamma basis is, as we know from before,

$$\mathbf{1},\ \gamma^\mu,\ \gamma^{\mu\nu},\ \gamma^\mu\gamma^5,\ \gamma^5. \quad (1.652)$$

The only object in this list having a non-zero trace is the unit matrix, $tr\mathbf{1} = 4$ since $tr\gamma^\mu = 0$ by the γ^5-trick, $tr\gamma^{\mu\nu} = 0$ by trace cyclicity, $tr\gamma^\mu\gamma^5, = 0$ again by the γ^5-trick and $tr\gamma^5 = 0$ by the γ^0-trick. With this information we get

$$tr(\gamma^\mu\gamma^\nu) = tr(g^{\mu\nu} + \gamma^{\mu\nu}) = 4g^{\mu\nu}, \quad (1.653)$$

$$tr(\gamma^\mu\gamma^\nu\gamma^\rho) = 0, \quad (1.654)$$

and finally

$$tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = tr((g^{\mu\nu} + \gamma^{\mu\nu})(g^{\rho\sigma} + \gamma^{\rho\sigma})) = tr(g^{\mu\nu}g^{\rho\sigma} + \gamma^{\mu\nu}g^{\rho\sigma} + g^{\rho\sigma}\gamma^{\rho\sigma} + \gamma^{\mu\nu}\gamma^{\rho\sigma}). \quad (1.655)$$

The only non-trivial term here is the last one: Using the idea explained previously (all terms in the expansion must have coefficient $+1$ or -1) we find the identity

$$\gamma^{\mu\nu}\gamma_{\rho\sigma} = \gamma^{\mu\nu}{}_{\rho\sigma} - 4\delta^{[\mu}_{[\rho}\gamma^{\nu]}{}_{\sigma]} - 2\delta^{\mu\nu}_{\rho\sigma}. \quad (1.656)$$

Thus

$$tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - (g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})). \quad (1.657)$$

Another type of useful identities are the following ones:

$$\gamma^\mu\gamma_\mu = 4,\ \ \gamma^\nu\gamma^\mu\gamma_\nu = -2\gamma^\mu,\ \ \gamma^\rho\gamma^\mu\gamma^\nu\gamma_\rho = \gamma^\rho(\gamma^{\mu\nu} + g^{\mu\nu})\gamma_\rho = 4g^{\mu\nu}. \quad (1.658)$$

A final useful identity of this type is

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu \quad (1.659)$$

where we should note the reversed order of the gamma matrices on the RHS.

We can now return to the trace expressions above. Consider again

$$tr\Big(\gamma^\mu(\not p' - m)\gamma^\nu(\not p + m)\Big) = tr\Big(\gamma^\mu\not p'\gamma^\nu\not p - m^2\gamma^\mu\gamma^\nu\Big), \quad (1.660)$$

where we have dropped the terms linear in m since they contain three gamma matrices and hence has zero trace. Explicitly this expression reads

$$= p'^{\rho}p^{\sigma}tr(\gamma^{\mu}\gamma_{\rho}\gamma^{\nu}\gamma_{\sigma}) - 4m^2 g^{\mu\nu} = 4p'^{\rho}p^{\sigma}(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - (g^{\mu\nu}g_{\rho\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho})) - 4m^2 g^{\mu\nu}. \tag{1.661}$$

Here we have first used $\gamma^{\mu}\gamma_{\rho} = \delta^{\mu}_{\rho} + \gamma^{\mu}{}_{\rho}$ and similarly for the other two gamma matrices, and then $tr(\gamma^{\mu}{}_{\rho}\gamma^{\nu}{}_{\sigma}) = -4(g^{\mu\nu}g_{\rho\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho})$. Simplifying this expression we get

$$= 4(p'^{\mu}p^{\nu} + p'^{\nu}p^{\mu} - p\cdot p' g^{\mu\nu} - m^2 g^{\mu\nu}). \tag{1.662}$$

The other trace gives the same result but with momenta k and k' instead. The whole answer for the unpolarised amplitude is therefore

$$|\mathcal{M}|^2_{unpol} = \frac{4e^4}{(q^2+i\epsilon)^2}(k_{\mu}k'_{\nu} + k_{\nu}k'_{\mu} - g_{\mu\nu}(k\cdot k' + m^2_{(\mu)}))(p^{\mu}p'^{\nu} + p^{\nu}p'^{\mu} - g^{\mu\nu}(p\cdot p' + m^2_{(e)})). \tag{1.663}$$

This result was discussed earlier in the course in the high energy limit where the electron mass could be neglected. Setting $m_{(e)} = 0$ above simplifies the result quite a bit:

$$|\mathcal{M}|^2_{unpol}|_{m_{(e)}=0} = \frac{8e^4}{(q^2+i\epsilon)^2}(k\cdot pk'\cdot p' + k\cdot p'k'\cdot p + m^2_{(\mu)}p\cdot p'). \tag{1.664}$$

To get out the physics from this result one must choose a frame where it can interpreted, e.g., the

1) The laboratory frame, i.e., fixed target, (Lab) or

2) the center of mass frame (CM).

We will do the analysis in the CM frame. Since the the two incoming particles is a particle/anti-particle pair they have the same mass and hence in this frame $\mathbf{p}_2 = -\mathbf{p}_1$. Then also the energies are the same which is also true for the out-going particle/anti-particle pair which however have different masses from the in-coming pair.

These aspects are part of what is called the **kinematics** of the process and its details depend heavily on the frame chosen. To be completely specific we have

$$p^{\mu}(e^-) = (E,0,0,p), \qquad p'^{\mu}(e^+) = (E,0,0,-p), \tag{1.665}$$

$$k^{\mu}(\mu^-) = (E,\mathbf{k}), \qquad k'^{\mu}(\mu^+) = (E,-\mathbf{k}), \tag{1.666}$$

where the in-coming particles move in the $\hat{z}$-direction and the out-going in some general but opposite directions $\mathbf{k}$ and $\mathbf{k}' = -\mathbf{k}$.

It then becomes an easy exercise to compute the various scalar products appearing in the expression for $|\mathcal{M}|^2$ obtained above: (recall that we have put $m_{(e^\pm)} = 0$ here which implies that $E = |p_z|$ for the electron/positron pair)

$$p \cdot p' = EE' - p_z p_z' = E^2 + E^2 = 2E^2, \qquad k \cdot k' = E^2 - \mathbf{k} \cdot \mathbf{k}' = E^2 + |\mathbf{k}|^2, \tag{1.667}$$

$$q^2 = (p + p')^2 = p^2 + p'^2 + 2p \cdot p' = 0 + 0 + 4E^2 = 4E^2, \tag{1.668}$$

$$p \cdot k = p' \cdot k' = (E, 0, 0, E) \cdot (E, \mathbf{k}) = E^2 - E|\mathbf{k}| \cos\theta, \tag{1.669}$$

$$p \cdot k' = p' \cdot k = (E, 0, 0, E) \cdot (E, -\mathbf{k}) = E^2 + E|\mathbf{k}| \cos\theta. \tag{1.670}$$

With these scalar products at hand we get

$$|\mathcal{M}|^2_{unpol} = \frac{e^4}{2E^4} \left((E^2 - E|\mathbf{k}| \cos\theta)^2 + (E^2 + E|\mathbf{k}| \cos\theta)^2 + 2E^2 m^2_{(\mu)} \right). \tag{1.671}$$

We note that the $|\mathbf{k}|$ terms cancel and hence , using $|\mathbf{k}|^2 = E^2 - m^2_{(\mu)}$, this simplifies to

$$|\mathcal{M}|^2_{unpol} = e^4 \left((1 + \frac{m^2_{(\mu)}}{E^2}) + (1 - \frac{m^2_{(\mu)}}{E^2}) \cos^2\theta \right). \tag{1.672}$$

The final task is to insert this into the differential cross section formula for $2 \to 2$ scattering in the CM frame which reads

$$(\frac{d\sigma}{d\Omega})_{CM}|_{unpol} = \frac{1}{2E_A 2E_B |v_A - v_B|} \frac{|\mathbf{p}_1|}{(2\pi)^2 4E_{CM}} |\mathcal{M}(p_A, p_B \to p_1, p_2)|^2. \tag{1.673}$$

To get the result in the case studied here we need to translate the notation in the formula to our present notation ($c = 1$ here, and $e^\pm$ are essentially massless)

$$E_A = E_B = E = \frac{1}{2} E_{CM}, \qquad |v_A - v_B| = 2, \qquad \mathbf{p}_1 = \mathbf{k} = \sqrt{E^2 - m^2_{(\mu)}}. \tag{1.674}$$

This gives

$$(\frac{d\sigma}{d\Omega})_{CM}|_{unpol,\ m_{(e)}=0} = \frac{1}{E^2_{CM}} \frac{1}{2} \frac{\sqrt{E^2 - m^2_{(\mu)}}}{(2\pi)^2 4E_{CM}} |\mathcal{M}(e^+e^- \to \mu^+\mu^-)|^2|_{unpol,\ m_{(e)}=0} =$$
$$= \frac{\alpha^2}{4E^2_{CM}} \sqrt{1 - \frac{m^2_{(\mu)}}{E^2}} \left((1 + \frac{m^2_{(\mu)}}{E^2}) + (1 - \frac{m^2_{(\mu)}}{E^2}) \cos^2\theta \right), \tag{1.675}$$

where $\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$ is the **fine structure constant**.

Integrating this over the angles produces the total cross section:

$$\sigma|_{unpol} = \int (\frac{d\sigma}{d\Omega})_{CM}|_{unpol} d\Omega = \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi (\frac{d\sigma}{d\Omega})_{CM}|_{unpol}$$
$$= \frac{4\pi\alpha^2}{3E^2_{CM}} \sqrt{1 - \frac{m^2_{(\mu)}}{E^2}} \left(1 + \frac{1}{2} \frac{m^2_{(\mu)}}{E^2} \right). \tag{1.676}$$

The square root in this expression is the *phase space* factor mentioned briefly above. This factor will play an important role next when we use this result for the cross section to draw some important physical conclusions.

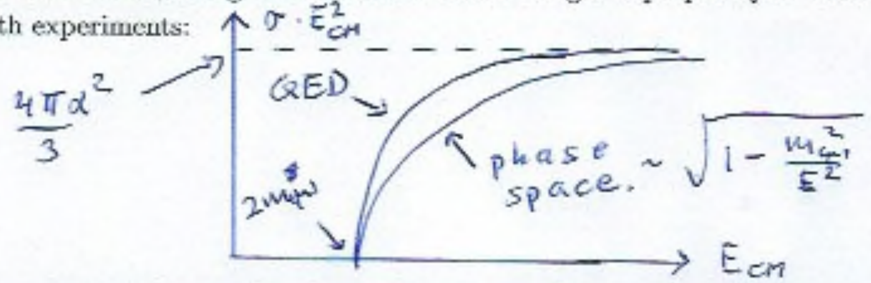
First we consider the high energy (i.e., $E >> m_{(\mu)}$) limit of these results:

$$(\frac{d\sigma}{d\Omega})_{CM}|_{unpol}(E \to \infty) \to \frac{\alpha^2}{4E_{CM}^2}(1+\cos^2\theta), \tag{1.677}$$

$$\sigma|_{unpol}(E \to \infty) \to \frac{4\pi\alpha^2}{3E_{CM}^2}\left(1 - \frac{3}{8}\left(\frac{m_{(\mu)}^2}{E^2}\right)^4 - ....\right), \tag{1.678}$$

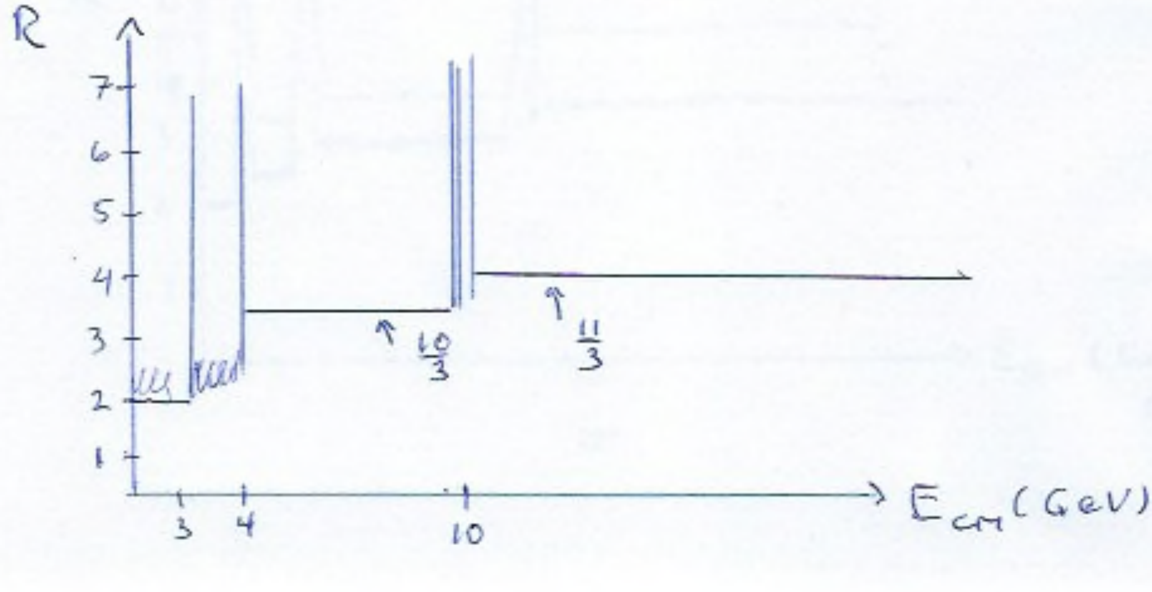
where the first formula was discussed in the first week of the course and in the second formula we recognise the first term for which we have kept the first non-zero correction.

There is clearly a **threshold** here: below $E_{CM} = 2m_{(\mu)}$ the cross section is zero which, however, is just a result of the phase space factor and is not a test of our QED calculation. The constant is nevertheless a bit interesting: It gives the correct result to about 10 per cent. On the other hand, using the full result above we get a proper QED result to be compared with experiments:



One can apply the above results also to the process $e^+e^- \to \tau^+\tau^-$ where the particles $\tau^\pm$ belong to the third family of leptons instead of the second as the $\mu^\pm$ particles. Looking at the above graph in this reveals that the mass of the τ is about 1800 MeV (about twice that of a proton) which should be compared to the mass of the μ which is about 100 MeV (which is $\approx$ 200 times the electron mass).

Applying the same formulas to processes where quarks are produced becomes then really interesting and some basic features of the standard model can inferred and understood. Define R as the $q\bar{q}$ cross section relative the $\mu^\pm$ one: $R := \frac{\sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)}$. Then the relevant graph for producing quark/anti-quark pairs is:



This graph contains a number of sharp peaks that we need to understand. They are due to various bound states that can be formed just below threshold for a new $q\bar{q}$ pair. Consider the previous graph for $\sigma(e^+e^- \to \mu^+\mu^-)$ and imaging increasing the energy of the $e^\pm$ pair from well below the threshold where muon-pair cannot be created. A bit before reaching the threshold the muon-pair can in fact be created if they end up in abound state since such a state has slightly lower energy than the energy of two muons at rest. After all, it will cost some energy to break up the bound state. The existence of this bound state, which is not contained in the graph computed here, will when the $e^\pm$ energy is high enough to create it, cause a sudden huge increase in the cross section which is seen as a peak in the graph above. Thus, such a peak will appear each time the energy is increased to a level where a new $q\bar{q}$ bound state can be created. Such bound states are very short-lived, e.g., the $\mu^+\mu^-$ muons of the dimuonium[38] state will annihilate in about $10^{-12}s$. The formation of bound states can be analysed in great detail by the methods developed here combined with a wave function description of the bound state itself. In QFT a bound state corresponds to a loop diagram which an infinite number of photon corrections. Section 5.2 in PS (not in the course) contains more details on this.

This graph can be explained easily by assuming the fundamental particles in Nature to have the following properties:

$$\text{Leptons}: \quad Q = -1: \quad e,\ \mu,\ \tau, \quad Q = 0: \quad \nu_e,\ \nu_\mu, \quad \nu_\tau, \tag{1.679}$$

$$\text{Quarks}: \quad Q = 2/3: \quad u,\ c,\ t, \quad Q = -1/3: \quad d,\ s, \quad b, \tag{1.680}$$

If the energy is high enough for quarks to be produced in the e^+e^- scattering the formula above for $\frac{d\sigma}{d\Omega}$ can still be used if the following changes are made:

1. For out-going states with charge Q: $e \to Qe$, i.e., $\frac{d\sigma}{d\Omega} \to Q^2\frac{d\sigma}{d\Omega}$.

2. Each quark has a new charge called **color** taking three values which are not observed in experimants: $\frac{d\sigma}{d\Omega} \to 3\frac{d\sigma}{d\Omega}$.

3. The produced quark/anti-quark ($q\bar{q}$) pairs are **confined** (more later): $q\bar{q} \to hadrons$. When produced the quark/anti-quark pairs are basically free since QCD is an **asymptotically free**[39] theory (will be discussed a bit more at the end of the course).

With this input we can compute the factor $3\Sigma_i Q_i^2$ which arises when the energy is raised step by step over a point where a new quark/anti-quark pair can be created:

a) i=u,d,s $\Rightarrow 3\Sigma_i Q_i^2 = 3((\frac{1}{3})^2 + (\frac{2}{3})^2 + (\frac{1}{3})^2) = 2$.

b) i=u,d,s,c $\Rightarrow 3\Sigma_i Q_i^2 = 2 + 3(\frac{2}{3})^2 = \frac{10}{3}$ for energies $> 2 \times 1.2 \times GeV$.

[38] See Jentschura et al, ArXiv physics/9706026, for a discussion.

[39] This discovery gave the 2004 Nobel prize to David Gross, David Politzer and Frank Wilczek.

c) i=u,d,s,c,b $\Rightarrow 3\Sigma_i Q_i^2 = \frac{10}{3} + 3(\frac{1}{3})^2 = \frac{11}{3}$ for energies $> 2 \times 4.7 \times GeV$

d) i=u,d,s,c,b,t=all $\Rightarrow 3\Sigma_i Q_i^2 = 3 \times 3((\frac{1}{3})^2 + (\frac{2}{3})^2) = 5$ for energies $> 2 \times 175 \times GeV$