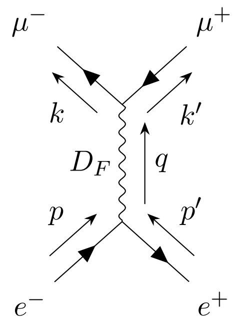
1.12.2 Helicity structure of polarised scattering processes: $e^+e^- \to \mu^+\mu^-$

Recall the scattering process $e^+e^- \to \mu^+\mu^-$ discussed in the previous section. The relevant Feynman diagram is



(1.681)

which we saw gave rise to the following expression by using the QED Feynman rules from the previous chapter:

$$i\mathcal{M}(e^+e^- \to \mu^+\mu^-) = \left(\bar{u}^r(k)(-ie\gamma^\mu)v^{r'}(k')\right)\left(\frac{-ig_{\mu\nu}}{q^2+i\epsilon}\right)\left(\bar{v}^{s'}(p')(-ie\gamma^\nu)u^s(p)\right)|_{q=p+p'}. \tag{1.682}$$

Note that the spinors u and v in the first factor are the same as the ones in the last factor except for masses $k^2 = m^2_{(\mu)}$ and $p^2 = m^2_{(e)}$, and the same for the primed momenta. Before squaring it, the scattering amplitude of course depends on the spin directions for the various particles given by the upper indices r, r', s, s' on the polarisation spinors u and v.

In this section we will be interested in computing the full polarised result for the differential cross section, that is, we need to square the scattering amplitude keeping the dependence of each particle's spin direction. To do this we need a little trick that will be explained below.

First we note that one can choose to quantise the spin for each particle independently, that is, in different directions. However, we will not do the complete calculation here but simplify it a bit by looking only at energies high enough that the masses of both the $e^\pm$ and the $\mu^\pm$ particles can be neglected. In this situation it is convenient to choose the quantisation direction to be along the motion of the particles e^- and the μ^- since *spin direction* can then be replaced by *helicity* which is a Lorentz invariant concept for massless particles.

To see this relation to helicity we consider the incoming $e^\pm$ which are moving in the positive and negative $\hat{z}$-direction, respectively for the e^- and e^+. The spinor for the electron is then, with s giving the polarisation in the $\hat{z}$ direction, i.e., $\xi^1 = \binom{1}{0}$ and $\xi^2 = \binom{0}{1}$,

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix}, \tag{1.683}$$

and since the e^- is moving in the positive $\hat{z}$-direction, we have $p^\mu = (E, 0, 0, p^3)$ with $p^3 > 0$ and hence

$$p \cdot \sigma := p^0 \mathbf{1} - \mathbf{p} \cdot \sigma = E\mathbf{1} - p^3 \sigma^3. \tag{1.684}$$

This implies that

$$\sqrt{p \cdot \sigma} = \sqrt{E - p^3}\frac{1}{2}(\mathbf{1} + \sigma^3) + \sqrt{E + p^3}\frac{1}{2}(\mathbf{1} - \sigma^3), \tag{1.685}$$

and similarly for $\sqrt{p \cdot \bar{\sigma}}$ but with the opposite sign in front of σ^3. For the u spinor this means in the high energy limit (in the positive $\hat{z}$-direction) where $p^\mu = (E, 0, 0, E)$

$$u^s(p) \to \sqrt{2E} \begin{pmatrix} \frac{1}{2}(\mathbf{1} - \sigma^3)\xi^s \\ \frac{1}{2}(\mathbf{1} + \sigma^3)\xi^s \end{pmatrix} = \sqrt{2E} \begin{pmatrix} \xi^2 \\ \xi^1 \end{pmatrix}. \tag{1.686}$$

Thus we find that the projections onto the L and R parts of this spinor $u^s(p)$ are related to the spin direction. If the electron in the scattering process we analyse here is chosen to have helicity $h = \frac{1}{2}$ it has a right-handed polarisation and hence described by the R part of u: (recall that ξ^1 is spin up in the $\hat{z}$-direction)

$$u^s_R(p) := P_R u^s(p) = \frac{1}{2}(1 + \gamma^5)u^s(p) = \sqrt{2E} \begin{pmatrix} 0 \\ \xi^1 \end{pmatrix} = \sqrt{2E} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{1.687}$$

Similarly for the left-handed helicity $h = -\frac{1}{2}$:

$$u^s_L(p) := P_L u^s(p) = \frac{1}{2}(1 - \gamma^5)u^s(p) = \sqrt{2E} \begin{pmatrix} \xi^2 \\ 0 \end{pmatrix} = \sqrt{2E} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{1.688}$$

These equations show how the chiral projections $P_L u^s(p)$ and $P_R u^s(p)$ are tied to the helicity in this massless high energy limit.

This makes it possible to project expressions like $\bar{v}\gamma^\mu u$ onto specific helicities (i.e., spin directions relative the direction of motion). For instance, for the incoming right-handed electron we use

$$\bar{v}^{s'}(p')\gamma^\mu u^s(p) \to \bar{v}^{s'}(p')\gamma^\mu P_R u^s(p) = \bar{v}^{s'}(p')\gamma^\mu \frac{1}{2}(1 + \gamma^5)u^s(p). \tag{1.689}$$

This result can be used in an interesting way by simply moving the projection operator P_R from the u spinor to the v spinor:

$$\begin{aligned} \bar{v}^{s'}(p')\gamma^\mu u^s_R(p) &= \bar{v}^{s'}(p')\gamma^\mu \frac{1}{2}(1 + \gamma^5)u^s(p) = \bar{v}^{s'}(p')\frac{1}{2}(1 - \gamma^5)\gamma^\mu u^s(p) \\ &= (\frac{1}{2}(1 + \gamma^5)v^{s'}(p'))^\dagger \gamma^0 \gamma^\mu u^s(p) = \bar{v}^{s'}_R(p')\gamma^\mu u^s(p), \end{aligned} \tag{1.690}$$

which tells us that a right-handed electron can only give a non-zero scattering amplitude if scattered against a left-handed positron. Note that $v_R^{s'}(p')$ is related to the ξ^1 which as we have shown in Chapter 3 for anti-particles is a left-handed positron. Thus

$$i\mathcal{M}(e_R^- e_L^+ \to \mu^- \mu^+) \neq 0, \text{ while } i\mathcal{M}(e_R^- e_R^+ \to \mu^- \mu^+) = 0. \tag{1.691}$$

This result, that was quoted already in Chap 1 of PS, is thus seen to be a result of the properties of the u and v spinors. Note that the total spin of the $e^\pm$ pair is zero for the case $e_R^- e_R^+$ and that, as we know from before, there is no spin zero component of the photon that this state can couple to. This explains why the corresponding scattering amplitude vanishes. There seems to be an issue here that requires an answer: the photon line connected to the each vertex propagates all components of the vector potential since it is just given by $-ig_{\mu\nu}/q^2$. The resolution of this problem is related to the so called Ward identity as we will see in the next lecture.

There is a second important implication of the result above about the projection operators and their connection to the spin direction: s can be left unspecified so one can compute the absolute square of $\bar{v}^{s'}(p')\gamma^\mu u_R^s(p)$ by summing over spin directions and using the completeness relations exactly as we did in the previous section for the unpolarised cross section. This is then done for both in-coming and out-going spin variables but without the factors of $\frac{1}{2}$ for the in-coming particles since this is not an average anymore.

To see how this works consider the square of the $e^\pm$-line factor above:

$$\begin{aligned} &\Sigma_{s,s'} \left(\bar{v}^{s'}(p')\gamma^\mu \frac{1+\gamma^5}{2} u^s(p) \right)^\dagger \left(\bar{v}^{s'}(p')\gamma^\nu \frac{1+\gamma^5}{2} u^s(p) \right) \\ &= \Sigma_{s,s'} \left(\bar{u}^s(p)\gamma^\mu \frac{1+\gamma^5}{2} v^{s'}(p') \right) \left(\bar{v}^{s'}(p')\gamma^\nu \frac{1+\gamma^5}{2} u^s(p) \right), \end{aligned} \tag{1.692}$$

where before taking the hermitian conjugate we have moved the projection operator $\frac{1+\gamma^5}{2}$ from the u spinor to the v spinor as explained above. This step does not change P_L to P_R since it passes two gamma-matrices and is hermitian. Thus the above result follows and we can use the completeness relation for v spinors. The last expression above then becomes

$$= \Sigma_s \left(\bar{u}^s(p)\gamma^\mu \frac{1+\gamma^5}{2} (\not{p}' - m)\gamma^\nu \frac{1+\gamma^5}{2} u^s(p) \right). \tag{1.693}$$

In the present high energy approximation we can neglect the mass term. Moving the $\bar{u}$ spinor to the far right of the expression and replace the $u\bar{u}$ with $\not{p} - m$ and dropping the mass term we get again a trace (as we did for the unpolarised case computed in the previous section)

$$= tr\left(\gamma^\mu \frac{1+\gamma^5}{2} \not{p}' \gamma^\nu \frac{1+\gamma^5}{2} \not{p} \right) = tr\left(\gamma^\mu \not{p}' \gamma^\nu \frac{1+\gamma^5}{2} \not{p} \right) = tr\left(\not{p}\gamma^\mu \not{p}' \gamma^\nu \frac{1+\gamma^5}{2} \right), \tag{1.694}$$

where we have moved the projection operators next to each other and used $P_L^2 = P_L$, and the trace cyclicity in the last step.

The reason for the last step was just to get it in a form that can be easily compared to the unpolarised calculation in the previous section. In fact, writing the above result as two terms as follows

$$= \frac{1}{2} p_\rho p'_\sigma tr(\gamma^\rho \gamma^\mu \gamma^\sigma \gamma^\nu) + \frac{1}{2} p_\rho p'_\sigma tr(\gamma^\rho \gamma^\mu \gamma^\sigma \gamma^\nu \gamma^5), \tag{1.695}$$

we see that, apart from the factor $\frac{1}{2}$ the first term is exactly the expression we evaluated in the previous section, while the second term is new.

To compute the second term we can use the gamma-basis and expand the four gamma matrices $\gamma^\rho \gamma^\mu \gamma^\sigma \gamma^\nu$ in terms of it. It will give gamma-basis elements with with four, two and zero vector indices. Then the trace gives a non-zero result only for the 4-index gamma term since that is related to γ^5. Thus the second term above becomes

$$tr(\gamma^\rho \gamma^\mu \gamma^\sigma \gamma^\nu \gamma^5) = -i\epsilon^{\rho\mu\sigma\nu} tr(\gamma^5 \gamma^5) = -4i\epsilon^{\rho\mu\sigma\nu}. \tag{1.696}$$

Recall the definition $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\epsilon^{0123} = +1$. The whole result for the trace is then

$$tr\left(\not{p}\gamma^\mu \not{p}'\gamma^\nu \frac{1+\gamma^5}{2}\right) = 2(p'^\mu p^\nu + p^\mu p'^\nu - g^{\mu\nu} p \cdot p' - i\epsilon^{\rho\mu\sigma\nu} p_\rho p'_\sigma). \tag{1.697}$$

Repeating these steps for the out-going fermion line we get

$$tr\left(\not{k}'\gamma_\mu \not{k}\gamma_\nu \frac{1+\gamma^5}{2}\right) = 2(k_\mu k'_\nu + k'_\mu k_\nu - g_{\mu\nu} k' \cdot k - i\epsilon_{\rho\mu\sigma\nu} k'^\rho k^\sigma). \tag{1.698}$$

The total result for the square of the scattering amplitude $|\mathcal{M}|^2$ is obtained by contracting together the last two expressions (noting that the indices $\rho\sigma$ must not be repeated):

$$|\mathcal{M}|^2|_{pol} = \frac{4e^4}{(q^2)^2}(2p \cdot k\, p' \cdot k' + 2p \cdot k'\, p' \cdot k - \epsilon^{\alpha\mu\beta\nu} \epsilon_{\rho\mu\sigma\nu} k'_\alpha k_\beta p^\rho p'^\sigma. \tag{1.699}$$

Here we need to expand the twice contraction product of two epsilon tensors:

$$\epsilon^{\alpha\mu\beta\nu} \epsilon_{\rho\mu\sigma\nu} = -4\delta^{\alpha\beta}_{\rho\sigma} := -2(\delta^\alpha_\rho \delta^\beta_\sigma - \delta^\alpha_\sigma \delta^\beta_\rho), \tag{1.700}$$

which follows from the fact that if also the last four indices are contracted pairwise we must get $\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma} = -24$.

Comment: In fact the uncontracted product gives

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta} = -4!\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}, \tag{1.701}$$

since if the 24 terms on the RHS are written out explicitly (all with coefficient 1 or -1) it gives the correct result for whatever choice of values of the eight indices one checks. The minus sign above comes from the Minkowski metric since by definition $\epsilon^{0123} = +1$ which, by lowering the indices with the metric, implies $\epsilon_{0123} = -1$.

Then we get for the last term

$$\epsilon^{\alpha\mu\beta\nu}\epsilon_{\rho\mu\sigma\nu}k'_\alpha k_\beta p^\rho p'^\sigma = -2(p\cdot k'\, p'\cdot k - p\cdot k\, p'\cdot k'), \tag{1.702}$$

which means that the last term cancels against a similar term in the rest of the expression. Thus, using the kinematic relations $q^2 = 4E^2$ and $p\cdot k' = p'\cdot k = E^2(1+\cos\theta)$, we get

$$|\mathcal{M}|^2|_{pol} = \frac{16e^4}{(q^2)^2}p\cdot k'\; p'\cdot k = e^4(1+\cos\theta)^2. \tag{1.703}$$

Finally, we have found that the high energy approximation of the polarised differential cross section for this process is (in the CM frame)

$$\left(\frac{d\sigma}{d\Omega}\right)(e^-_R e^+_L \to \mu^-_R\mu^+_L)|_{CM,\,pol} = \frac{\alpha^2}{4E^2_{CM}}(1+\cos\theta)^2, \tag{1.704}$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine structure constant.

Similarly, we get for the same process but with reversed helicities for the out-going particles

$$\left(\frac{d\sigma}{d\Omega}\right)(e^-_R e^+_L \to \mu^-_L\mu^+_R)|_{CM,\,pol} = \frac{\alpha^2}{4E^2_{CM}}(1-\cos\theta)^2, \tag{1.705}$$

which can be understood to arise from letting the μ^- in the last case exit in the direction of the μ^+ in the previous case (computed above) but keeping its spin direction. This will flip the helicities for both out-going particles.

Then using the fact that QED is parity invariant we may flip all four helicities simultaneously in both the results above which gives

$$\left(\frac{d\sigma}{d\Omega}\right)(e^-_R e^+_L \to \mu^-_R\mu^+_L)|_{CM,\,pol} = \left(\frac{d\sigma}{d\Omega}\right)(e^-_L e^+_R \to \mu^-_L\mu^+_R)|_{CM,\,pol}, \tag{1.706}$$

and

$$\left(\frac{d\sigma}{d\Omega}\right)(e^-_R e^+_L \to \mu^-_L\mu^+_R)|_{CM,\,pol} = \left(\frac{d\sigma}{d\Omega}\right)(e^-_L e^+_R \to \mu^-_R\mu^+_L)|_{CM,\,pol}. \tag{1.707}$$

We can now check that by summing and averaging over out-going and in-coming helicities, respectively, we do obtain the same result as in the previous section:

$$\left(\frac{d\sigma}{d\Omega}\right)(e^-e^+ \to \mu^-\mu^+)|_{CM,\,unpol} = (\frac{1}{2})^2\Sigma_{all\,four\,pol\,results\,above} = \frac{\alpha^2}{4E^2_{CM}}(1+\cos^2\theta). \tag{1.708}$$

To summarise, we have found above that the trick of inserting projection operators to make it possible to sum over spin directions and thus use completeness relations also worked for the out-going μ particles. This fact follows since the u and v spinors used for the μ particles are valid for general momenta. They were shown in Chapter 3 to be derivable by a Lorentz transformation from the rest frame. To demonstrate even more explicitly that these u and v spinors are the correct ones to use we can derive $\mathcal{M}$ directly in this high energy limit and verify this fact.

To do this we return to (1.687) above

$$u_R^s(p) := P_R u^s(p) = \frac{1}{2}(1+\gamma^5)u^s(p) = \sqrt{2E}\begin{pmatrix} 0 \\ \xi^1 \end{pmatrix} = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{1.709}$$

To compute $\mathcal{M}(e_R^- e_L^+ \to \mu_R^- \mu_L^+)$ we need $\bar{v}_R^{s'}(p')\gamma^\mu P_R u^s(p)$ and thus the explicit form of v_R^s:

$$v_R^s(p) := P_R v^s(p) = \frac{1}{2}(1+\gamma^5)v^s(p) = \sqrt{2E}\begin{pmatrix} 0 \\ -\xi^2 \end{pmatrix} = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}. \tag{1.710}$$

The reason for this form of the v spinor is that the e^+ particles is left-handed and moves in the negative $\hat{z}$-direction and hence its spin is pointing in the positive $\hat{z}$-direction (i.e., the two $e^\pm$ particles have $s = +1$ together). Since this is an anti-particle it is therefore described by ξ^2 instead of ξ^1. The minus sign in $v_R^s(p)$ above appears already in the definition of the v spinor.

Using these results we see that

$$\bar{v}_R^{s'}(p')\gamma^\mu P_R u^s(p) = 2E(0,0,0,-1)\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 2E(0,-1)\sigma^\mu\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{1.711}$$

Thus the Lorentz *vector* that is the result of this calculation reads (as in Chap 1)

$$\bar{v}_R^{s'}(p')\gamma^\mu P_R u^s(p) = -2E\begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}. \tag{1.712}$$

It is now possible to perform an active rotation of this Lorentz vector around the $\hat{y}$-direction to make it lie in the xz-plane pointing in the direction making an angle θ with the z axis. This space rotation is given by the following Lorentz matrix

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}. \tag{1.713}$$

Applying this to the previous Lorentz vector we get its form when rotated to the θ direction as appropriate for the out-going μ^- particle:

$$\begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ \cos\theta \\ i \\ -\sin\theta \end{pmatrix}. \tag{1.714}$$

In the expression for $i\mathcal{M}$ derived above we see that we need

$$\bar{u}(k)\gamma^\mu v(k') = (\bar{v}(k')\gamma^\mu u(k))^\star = -2E \begin{pmatrix} 0 \\ \cos\theta \\ -i \\ -\sin\theta \end{pmatrix}, \tag{1.715}$$

which implies that

$$\mathcal{M}(e_R^- e_L^+ \to \mu_R^- \mu_L^+) = \frac{e^2}{q^2}(\bar{v}(p')\gamma^\mu u(p)) g_{\mu\nu} (\bar{u}(k)\gamma^\nu v(k')) = -4E^2 \frac{e^2}{q^2}(1+\cos\theta), \tag{1.716}$$

which after using the kinematic relations becomes

$$\mathcal{M}(e_R^- e_L^+ \to \mu_R^- \mu_L^+) = -e^2(1+\cos\theta). \tag{1.717}$$

We therefore find the same $|\mathcal{M}|^2$ as when we used the projection operator trick above.

Comment: The manipulations with the projection operators above and the results , i.e., moving them from the u spinor to the v spinor and find that they stay the same, have a direct analogue in terms of the QED Lagrangian. Let us consider the possibility for a spin 1/2 particle described by a Dirac, Weyl or Majorana spinor, to have electric charge and/or non-zero mass. To answer this we have to check if the interaction term and mass term can exist in each of these cases. In the case of Dirac spinors we know that both exist (i.e., are not identically zero):

$$\text{Dirac}: \quad \mathcal{L}_{int} = -e\bar{\psi}\gamma^\mu\psi A_\mu \neq 0, \quad \mathcal{L}_{mass} = m\bar{\psi}\psi \neq 0. \tag{1.718}$$

For Weyl spinors this is no longer the case. Here we refer to a situation where only one of ψ_L or ψ_R exist[40]

$$\text{Weyl}: \quad \mathcal{L}_{int} = -e\bar{\psi}\gamma^\mu\psi A_\mu \neq 0, \quad \mathcal{L}_{mass} = m\bar{\psi}\psi = 0. \tag{1.719}$$

This is clear since as we saw in the computation of the polarised cross section above the interaction term is non-zero. The mass term term on the other hand couples always ψ_L to ψ_R and this vanishes if only one of the appear in the theory. Finally, for Majorana spinors we have

$$\text{Majorana}: \quad \mathcal{L}_{int} = -e\bar{\psi}\gamma^\mu\psi A_\mu = 0, \quad \mathcal{L}_{mass} = m\bar{\psi}\psi \neq 0. \tag{1.720}$$

The first result follows from the fact that $C\gamma^\mu$ is a symmetric matrix.
Exercise: Show this!

[40] Or that they are in different representations of some gauge group so that the product $\psi_L\psi_R$ cannot be used in the Lagrangian in either of the two terms we consider here.