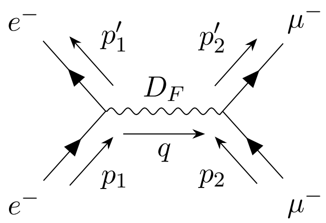
1.12.4 Crossing symmetry

To understand what *crossing symmetry* is we consider first the scattering $e^-\mu^- \to e^-\mu^-$ which we will later compare to the one discussed above, that is $e^-e^+ \to \mu^-\mu^+$. Since both of these come from the same second order term in the QED perturbation theory, that is $\frac{1}{2}\int d^4z_1\mathcal{H}_I(z_1)\int d^4z_2\mathcal{H}_I(z_2)$, there should be some kind of relation between their scattering amplitudes.

So let us start by considering the process $e^-\mu^- \to e^-\mu^-$ which is given by the Feynman diagram



. (1.721)

Note that this diagram is quite different from the one we have discussed so far for the scattering $e^-e^+ \to \mu^-\mu^+$. Recall that time is pointing upwards in these Feynman diagrams. Using the Feynman rules in QED, this diagram gives the scattering amplitude

$$
\begin{aligned}
&i\mathcal{M}(e^-\mu^- \to e^-\mu^-) \\
&= i\frac{e^2}{q^2}\left(\bar{u}^{s_1'}(p_1')\gamma^\mu u^{s_1}(p_1)\right)\left(\bar{u}^{s_2'}(p_2')\gamma_\mu u^{s_2}(p_2)\right)|_{q=p_1-p_1'}.
\end{aligned} \tag{1.722}
$$

The unpolarised $|\mathcal{M}|^2$ obtained from this expression is

$$
\begin{aligned}
&|\mathcal{M}(e^-\mu^- \to e^-\mu^-)|^2_{unpol} \\
&= \frac{e^4}{4(q^2)^2}tr\left((\not{p}_1' + m_{(e)})\gamma^\mu(\not{p}_1 + m_{(e)})\gamma^\nu\right)tr\left((\not{p}_2' + m_{(\mu)})\gamma_\mu(\not{p}_2 + m_{(\mu)})\gamma_\nu\right).
\end{aligned} \tag{1.723}
$$

This result can be compared to $|\mathcal{M}(e^-e^+ \to \mu^-\mu^+)|^2_{unpol}$ that we have computed previously

$$
\begin{aligned}
&|\mathcal{M}(e^-e^+ \to \mu^-\mu^+)|^2_{unpol} \\
&= \frac{e^4}{4(q^2+i\epsilon)^2}tr\Big(\gamma_\mu(\not{k} + m_{(\mu)})\gamma_\nu(\not{k}' - m_{(\mu)})\Big)tr\Big(\gamma^\mu(\not{p}' - m_{(e)})\gamma^\nu(\not{p} + m_{(e)})\Big)
\end{aligned} \tag{1.724}
$$

These two expressions are exactly equal to each other if the following substitution is made:

$$
p \to p_1 \quad p' \to -p_1' \quad k \to p_2' \quad k' \to -p_2. \tag{1.725}
$$

This is a nice observation since it may save us the work needed to do the gamma traces for the new diagram above: we can just use the substitution directly in the final result for $|\mathcal{M}(e^-e^+ \to \mu^-\mu^+)|^2_{unpol}$ and obtain the result for $|\mathcal{M}(e^-\mu^- \to e^-\mu^-)|^2_{unpol}$ after the traces are done.

However, the kinematics for $e^-\mu^- \to e^-\mu^-$ is entirely different from the previous case $e^-e^+ \to \mu^-\mu^+$ so this must be analysed again (it does not follow from crossing symmetry). In the center of mass (CM) system we now have for the in-coming particles (again in the high energy limit where we can set $m_{(e)} = 0$) with$k > 0$

$$p_1^\mu = (k, k\hat{\mathbf{z}}), \quad p_2^\mu = (E, -k\hat{z}), \quad \text{where } E = \sqrt{\mathbf{k}^2 + m_{(\mu)}^2}. \tag{1.726}$$

For the out-going particles we have, with $\mathbf{k}$ in some general direction

$$p'^\mu_1 = (k, \mathbf{k}), \quad p'^\mu_2 = (E, -\mathbf{k}), \quad \text{where } E = \sqrt{\mathbf{k}^2 + m_{(\mu)}^2}. \tag{1.727}$$

Note that E and $k = |\mathbf{k}|$ are the same for both in-coming and out-going momenta in the CM frame.

These momenta give directly (using the notation $k := \mathbf{k}$ and with $\mathbf{k} \cdot \hat{z} = k\cos\theta$)

$$p_1 \cdot p_2 = p'_1 \cdot p'_2 = kE + k^2, \quad p_1 \cdot p'_2 = p'_1 \cdot p_2 = kE + k^2\cos\theta, \quad p_1 \cdot p'_1 = k^2 - k^2\cos\theta, \quad p_2 \cdot p'_2 = E^2 - k^2\cos\theta. \tag{1.728}$$

It is perhaps even more interesting to check what q^2 is. In the limit $m_{(e)} = 0$ we find:

$$q^2 = (p_1 - p'_1)^2 = p_1^2 + p_1'^2 - 2p_1 \cdot p'_1 = 0 + 0 - 2p_1 \cdot p'_1 = -2k^2(1 - \cos\theta). \tag{1.729}$$

Doing the traces over gamma matrices this gives the following $|\mathcal{M}|^2$ for the $e^-\mu^-$ scattering, for the unpolarised case,

$$|\mathcal{M}|^2|_{unpol} = \frac{2e^2}{k^2(1-\cos\theta)^2}\Big((E+k)^2 + (E + k\cos\theta)^2 - m_{[\mu)}^2(1 - \cos\theta)\Big). \tag{1.730}$$

Inserting this into the general differential cross section formula for $2 \to 2$ scattering

$$(\frac{d\sigma}{d\Omega})|_{CM} = \tag{1.731}$$

Using the following translation

$$E_A \to k, \;\; E_B \to E, \;\; |\mathbf{p_1}| \to k, \;\; |v_A - v_B| = |\frac{k_z^A}{E_A} - \frac{k_z^B}{E_B}| \to |\frac{k}{k} + \frac{k}{E}| = 1 + \frac{k}{E}, \tag{1.732}$$

the differential cross section in our $e^-\mu^- \to e^-\mu^-$ case becomes

$$(\frac{d\sigma}{d\Omega})|_{CM} = \frac{1}{2k2E}\frac{1}{1+\frac{k}{E}}\frac{k}{(2\pi)^2 4(E+k)}|\mathcal{M}(e^-\mu^- \to e^-\mu^-)|^2_{unpol} \tag{1.733}$$

$$= \frac{\alpha^2}{2k^2(E+k)^2(1-\cos\theta)^2}\Big((E+k)^2 + (E + k\cos\theta)^2 - m_{(\mu)}^2(1-\cos\theta)\Big). \tag{1.734}$$

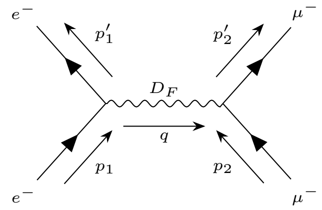
For enough high energies where also $m_{(\mu)}^2$ can be neglected, $E_{CM} = 2k$ and this result reduces to

$$\frac{d\sigma}{d\Omega}(e^-\mu^- \to e^-\mu^-)|_{CM, E>>m_{(\mu)}} = \frac{\alpha^2}{2E_{CM}^2(1-\cos\theta)^2}\left(4 + (1+\cos\theta)^2\right) \tag{1.735}$$

It is interesting to look at this result close to the forward direction, i.e., as $\theta \to 0$: using $\cos\theta = 1 - \frac{1}{2}\theta^2 + ...$ we get

$$\frac{d\sigma}{d\Omega}(e^-\mu^- \to e^-\mu^-)|_{CM} \to \frac{\alpha^2}{2E_{CM}^2 \frac{1}{4}\theta^4}(4+4) = \frac{16\alpha^2}{E_{CM}^2}\frac{1}{\theta^4}. \tag{1.736}$$

This result is characteristic of Coulomb scattering (i.e., elastic scattering involving only electric forces) and comes from the special kinematics of this process:



$$\propto \frac{1}{q^2} \Rightarrow (\frac{d\sigma}{d\Omega})_{CM} \propto \frac{1}{(q^2)^2}, \tag{1.737}$$

together with the kinematics relation

$$q^2 = -2p_1 \cdot p_1' = -2k^2(1-\cos\theta) \approx -k^2\theta^2. \tag{1.738}$$

The physics here can be expressed as the virtual photon with momentum q is almost on-shell close to the forward direction $\theta = 0$ where the differential cross section goes as $\frac{1}{\theta^4}$. This is known in physics since 1911 as Rutherford scattering.

After having discussed the two different processes $e^-\mu^- \to e^-\mu^-$ and $e^-e^+ \to \mu^-\mu^+$ and there crossing relation we should now try to understand the origin of this crossing symmetry. Consider a general $AB \to 12$ process in QED which from first principles is given in perturbation theory by (note the particle type associated to the current $j^\mu = \bar{\psi}\gamma^\mu\psi$ in each vertex)

$$\langle 12|T\left(\frac{1}{2}(-ie\int d^2z_1\bar{\psi}_{(e)}\gamma^\mu\psi_{(e)}A_\mu)(-ie\int d^2z_2\bar{\psi}_{(\mu)}\gamma^\nu\psi_{(\mu)}A_\nu)\right)|AB\rangle, \tag{1.739}$$

where the contraction of the two vector potentials gives the photon propagator which will not play a role in the discussion below. Assume now that particle A is the in-coming electron that appears in both the processes we are looking at here. Thus

$$|A\rangle = \sqrt{2E_A}a^{\dagger s}_{\mathbf{p}_A}|0\rangle. \tag{1.740}$$

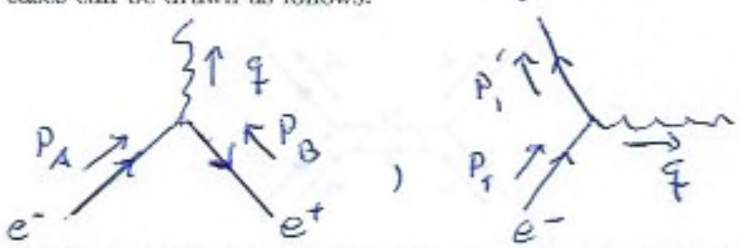
If we now want to find the possible ways to contract the interaction term above for the electron with this in-coming state $|A\rangle$ there are two ways to do it. They correspond to the two directions the electron fermion-line in the Feynman diagram can take: Either is continues "backwards" and contracts with an in-coming positron as in the e^-e^+ scattering case , or in contracts with an electron in the out-going state as in the $e^-\mu^-$ scattering.

These possibilities follow directly from splitting ψ into $+$ and $-$ parts: at the interaction point z_1 we have

$$(\bar{\psi}^+\psi^+ + \bar{\psi}^-\psi^- + \bar{\psi}^+\psi^- + \bar{\psi}^-\psi^+)|A\rangle. \tag{1.741}$$

Remembering that $|A\rangle \propto a^\dagger|0\rangle$ and ψ contains a and $b^\dagger$ while $\bar{\psi}$ contains b and $a^\dagger$, there are two possible contractions involving $|A\rangle$, namely with $\psi^+(a)$ in the first term and with $\psi^+(a)$ in the last term. Thus this in-coming line starting with the electron state $|A\rangle$ can continue either from the first term with $\bar{\psi}^+$ contracting with an in-coming positron (since $\bar{\psi}^+$ contains only b) or from the last term with $\bar{\psi}^-$ contracting with an out-going electron (since $\bar{\psi}^-$ contains only $a^\dagger$).

The two cases can be drawn as follows:



Squaring the fermionic lines in these two diagrams leads to the following factors in $|\mathcal{M}|^2$:

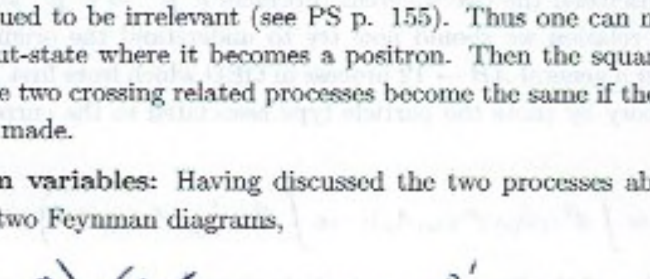
$$\Sigma_s v^s \bar{v}^s = \not{p}_B - m, \quad \Sigma_s u^s \bar{u}^s = \not{p}_1' + m. \tag{1.742}$$

But if q is the same in the two cases and the electron momentum also the same ($p_A = p_1$) then $p_B = -p_1'$ and hence

$$\not{p}_B - m = -(\not{p}_1' + m) \Rightarrow \Sigma u\bar{u} = -\Sigma v\bar{v}, \tag{1.743}$$

which explains the crossing symmetry relation between the two diagrams. The minus sign above can be argued to be irrelevant (see PS p. 155). Thus one can move the in-coming electron to the out-state where it becomes a positron. Then the square of the scattering amplitudes for the two crossing related processes become the same if the above momentum substitutions are made.

Mandelstam variables: Having discussed the two processes above and noted the difference in the two Feynman diagrams,



this difference can be expressed simply by giving the photon momentum and its relation to the in and out states:

$$e^+e^- : \ q = p_1 + p_2, \quad e^-\mu^- : \ q = p_1 - p_1', \tag{1.744}$$

where we have renamed the in-coming momenta in the first case to conform with the other cases: $p \to p_1$ and $p' \to p_2$.

If we consider instead e^-e^- scattering there is also a third kind of Feynman diagram where the two out-going legs are crossing each other which leads to the photon momentum

$$e^-e^- : \quad q = p_1 - p_2'. \tag{1.745}$$

These three cases are usually referred to the s, t, and u channel, respectively, and expressed in terms of the **Mandelstam variables** with the same names:

$$s := (p_1 + p_2)^2, \quad t := (p_1 - p_1')^2, \quad u := (p_1 - p_2')^2. \tag{1.746}$$

The virtue of these definitions is that crossing symmetry can be used simply by swapping some of these variables for others. An example is provided by the diagrams we discussed above. In terms of the Mandelstam variables we find

$$|\mathcal{M}(e^-e^+)|^2_{unpol} = \frac{8e^4}{s^2}\left((\frac{t}{2})^2 + (\frac{u}{2})^2\right), \tag{1.747}$$

while

$$|\mathcal{M}(e^-\mu^-)|^2_{unpol} = \frac{8e^4}{t^2}\left((\frac{s}{2})^2 + (\frac{u}{2})^2\right). \tag{1.748}$$

Another useful feature of these Mandelstam variables is their sum: (renaming the momenta as p_1, p_2 for the in-coming and p_3, p_4 for the out-going and using $p_i^2 = m_i^2$ for $i = 1.2.3.4$)

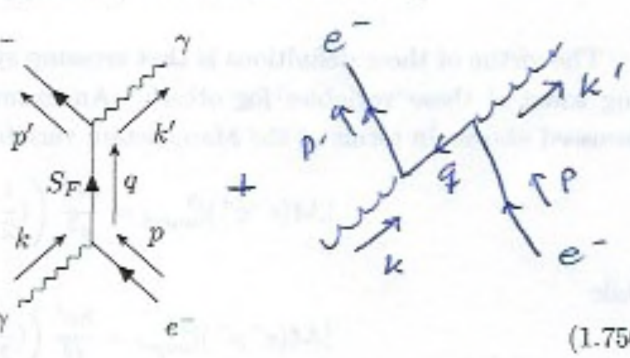
$$\begin{aligned} s+t+u = (p_1+p_2)^2 + (p_1-p_3)^2 + (p_1-p_4)^2 &= \Sigma_i m_i^2 + 2p_1 \cdot p_2 + p_1^2 - 2p_1 \cdot p_3 + p_1^2 - 2p_1 \cdot p_4 \\ &= \Sigma_i m_i^2 + 2p_1(p_2 - p_3 - p_4) + 2p_1^2 = \Sigma_i m_i^2, \end{aligned} \tag{1.749}$$

where we have used momentum conservation in the last step.

1.12.5 Compton scattering, photon polarisation sums and Ward identities

In this section we will compute the cross section for Compton scattering which involves photons in the in-coming and out-going states. This fact leads to deep issues concerning polarisation sums and Lorentz invariance when the scattering amplitude is squared to obtain the cross section. The outcome of this discussion is the need for introducing **Ward identities**.

This process requires two Feynman diagrams: (note the reason for this)



(1.750)

which gives

$$i\mathcal{M} = \epsilon^*_\nu(k')\bar{u}^{s'}(p')(-ie\gamma^\nu)\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2}(-ie\gamma^\mu)u^s(p)\epsilon_\mu(k)$$

$$+\,\epsilon_\mu(k)\bar{u}^{s'}(p')(-ie\gamma^\mu)\frac{i(\not{p}-\not{k}'+m)}{(p-k')^2-m^2}(-ie\gamma^\nu)u^s(p)\epsilon^*_\nu(k') =$$

$$= -ie^2\epsilon_\mu(k)\epsilon^*_\nu(k')\bar{u}^{s'}(p')\left(\frac{\gamma^\nu(\not{p}+\not{k}+m)\gamma^\mu}{(p+k)^2-m^2}+\frac{\gamma^\mu(\not{p}-\not{k}'+m)\gamma^\nu}{(p-k')^2-m^2}\right)u^s(p). \qquad (1.751)$$

We will try to square this below but first we note that this expression can be simplified somewhat by using the properties of the momenta, namely

$$k^2 = k'^2 = 0, \quad p^2 = p'^2 = m^2, \quad (\not{p}-m)u(p) = 0. \qquad (1.752)$$

Then we get

$$(p+k)^2 - m^2 = p^2 + k^2 + 2p\cdot k - m^2 = 2p\cdot k = 2p'\cdot k', \qquad (1.753)$$

$$(p-k')^2 - m^2 = p^2 + k'^2 - 2p\cdot k - m^2 = -2p\cdot k' = -2p'\cdot k, \qquad (1.754)$$

and

$$(\not{p}+\not{k}+m)\gamma^\mu u(p) = (-\gamma^\mu\not{p}+\{\not{p},\gamma^\mu\}+\not{k}\gamma^\mu+m\gamma^\mu)u(p) = \not{k}\gamma^\mu u(p)+2p^\mu u(p). \quad (1.755)$$

The expression in the second term can be rewritten in a similar way giving

$$i\mathcal{M} = -ie^2\epsilon_\mu(k)\epsilon^*_\nu(k')\bar{u}^{s'}(p')\left(\frac{\gamma^\nu\not{k}\gamma^\mu+2\gamma^\nu p^\mu}{2p\cdot k}+\frac{\gamma^\mu\not{k}'\gamma^\nu-2\gamma^\mu p^\nu}{2p\cdot k'}\right)u^s(p). \qquad (1.756)$$

Now we have to face the issue of squaring this expression to get the cross section. The fermionic part of it we are by now familiar with and we know how to compute its absolute square using the completeness relations for u^s and v^s (polarisation) spinors. However, also the photon polarisation tensors in the above expression for $i\mathcal{M}$ must be handled and their absolute square computed. The problem is that ϵ_μ is really ϵ_μ^λ where the index λ runs over the number of independent degrees of freedom of the photon, i.e., $\lambda = +, -$ for the two circular polarised states of the photon. The polarisation tensor (which is a vector here) ϵ_μ^λ is thus not a square matrix and hence one cannot write down a completeness relation for it! This issue is a direct effect of the gauge invariance of the Maxwell theory.

As we have discussed before gauge invariance makes it possible to choose the Lorentz gauge $\partial_\mu A^\mu = 0$ as a condition on the vector potential A_μ. Using this condition the Maxwell's equation becomes a $\Box$ equation. Also the remaining gauge transformation parameter after imposing the Lorentz condition is given by the solutions of $\Box\Lambda = 0$, that is, waves. Thus we can restrict the vector potential one step further by e.g. choosing the additional condition $A_0 = 0$. Then the Lorentz condition reduces to $\nabla \cdot \mathbf{A} = 0$ known as the Coulomb gauge. Note that the Feynman propagator is derived in the Lorentz gauge and therefore propagates all four components of A_μ.

In momentum space this means that the polarisation tensor for the vector potential will satisfy, in the Coulomb gauge,

$$\epsilon_\mu^\lambda(p) = (0, \boldsymbol{\epsilon}^\lambda), \text{ where } \mathbf{p} \cdot \boldsymbol{\epsilon}^\lambda = 0. \tag{1.757}$$

Clearly the vector potential will only contain two independent degrees of freedom so λ takes on only two values, e.g., $+, -$ for right and left polarised photons. This becomes clear if the photon is traveling in the positive $\hat{z}$-direction. Then the two circular polarisations are given by $\epsilon_\mu^\pm = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$. This three-dimensional $\boldsymbol{\epsilon}^\lambda$ is normalised $\boldsymbol{\epsilon}^{\lambda'} \cdot \boldsymbol{\epsilon}^\lambda = \delta^{\lambda\lambda'}$.

These two $\boldsymbol{\epsilon}^\lambda$ vectors can easily be extended by adding a third one, also normalised to one, $\boldsymbol{\epsilon}^3 := \frac{\mathbf{p}}{|\mathbf{p}|}$. This way we obtain a completeness relation in the three space directions $(i, j = 1, 2, 3)$

$$\Sigma_{\alpha=+,-,3} (\epsilon_i^\alpha)^\star \epsilon_j^\alpha = \delta_{ij}. \tag{1.758}$$

This completeness relation can then be used in the space part of the above computation of the square $|\mathcal{M}|^2$ if we write it as

$$\Sigma_{\lambda=+,-} (\epsilon_i^\lambda)^\star \epsilon_j^\lambda = \delta_{ij} - \frac{p_i p_j}{|\mathbf{p}|^2}. \tag{1.759}$$

Finally, we must now extend this completeness relation valid in space to the whole of Minkowski, i.e., add also a time-like vector ϵ_μ^0. However, this is not possible if we want to find a Lorentz covariant completeness relation of the kind

$$\Sigma_{A=(0,+,-,3)} (\epsilon_\mu^A)^\star \epsilon_\nu^A = -g_{\mu\nu}. \tag{1.760}$$

The reason this is impossible is that for this relation to work the added time-like vector must have a negative absolute square!

To be able to discuss this in terms of scattering amplitudes we express a general Feynman diagram with an external out-going photon with polarisation λ as

$$i\mathcal{M} := i\mathcal{M}^{\mu}\epsilon_{\mu}^{\lambda\star}. \tag{1.761}$$

Squaring this to get the unpolarised cross section means that we sum over polarisations which gives

$$\Sigma_{\lambda}|\mathcal{M}|^2 := \Sigma_{\lambda}|\mathcal{M}^{\mu}\epsilon_{\mu}^{\lambda\star}|^2 = \Sigma_{\lambda}\epsilon_{\mu}^{\lambda\star}\epsilon_{\nu}^{\lambda}\mathcal{M}^{\mu\star}\mathcal{M}^{\nu}. \tag{1.762}$$

With the λ sum running over two values (in the xy plane as above for photons moving in the z-direction) this becomes

$$\Sigma_{\lambda}|\mathcal{M}|^2 = |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2. \tag{1.763}$$

This result is correct but not Lorentz covariant which makes it less useful. In order to keep manifest Lorentz invariance we must insist on doing the **replacement**

$$\Sigma_{\lambda=+,-}\epsilon_{\mu}^{\lambda\star}\epsilon_{\nu}^{\lambda} \to -g_{\mu\nu}. \tag{1.764}$$

If we do this we must check what goes wrong: clearly now

$$\Sigma_{\lambda}|\mathcal{M}|^2 = |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 \to -g_{\mu\nu}\mathcal{M}^{\mu\star}(k)\mathcal{M}^{\nu}(k) = -|\mathcal{M}^0|^2 + |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 + |\mathcal{M}^3|^2, \tag{1.765}$$

so we now have two terms that are unphysical and unwanted, the 0 and 3 component terms. The first one is even appearing with a minus sign which is a disaster for unitarity.

However, there is a way out of this problem. Consider again $\mathcal{M}^{\mu}$ and note that the μ is connected to the photon line leaving the Feynman diagram from a fermion line due to the interaction term in QED:

$$\mathcal{L}_{int} = -e\bar{\psi}\gamma^{\mu}\psi A_{\mu} := -ej^{\mu}A_{\mu}. \tag{1.766}$$

This means that, doing all the contractions except the ones involving the spinors in the current of the z_1 vertex, $j^{\mu} = \bar{\psi}\gamma^{\mu}\psi$,

$$\mathcal{M}^{\mu} = \int d^4z e^{ik\cdot z}\langle f|j^{\mu}|i\rangle, \tag{1.767}$$

where i refers to all the in-going particles and f to all the out-going ones. The conservation of the current $\partial_{\mu}j^{\mu} = 0$ in the Fourier transformed version becomes $k_{\mu}j^{\mu} = 0$. Hence each external photon line must satisfy the

$$\text{Ward identity}: \quad k_{\mu}\mathcal{M}^{\mu}(k) = 0. \tag{1.768}$$

This is usually expressed by saying that if one replaces any photon polarisation tensor in a scattering amplitude by its momentum, i.e., $\epsilon_{\mu}(k)\mathcal{M}^{\mu} \to k_{\mu}(k)\mathcal{M}^{\mu}$ it must vanish.

That the Ward identity solves the problem above with the unwanted $|\mathcal{M}|^2$ terms follows since (again with the photon moving in the z-direction, $k^{\mu} = (k, 0, 0, k)$)

$$k_{\mu}\mathcal{M}^{\mu}(k) = k\mathcal{M}^0 - k\mathcal{M}^3 = 0 \Rightarrow \mathcal{M}^0 = \mathcal{M}^3, \tag{1.769}$$

which implies that the two unwanted terms cancel each other out in the covariant $|\mathcal{M}^\mu|^2$ above!

To see the Ward identity at work we can just look at the interaction vertex itself. If we consider the case with an out-going photon and an in-going and out-going electron we have from the Feynman rules

$$i\mathcal{M} = \epsilon_{\mu\star}(k)\bar{u}^{s'}(p')(-i\gamma^\mu)u^s(p)|_{k=p-p'}. \tag{1.770}$$

To check the Ward identity we must therefore compute

$$\begin{aligned} k_\mu(k)\bar{u}^{s'}(p')(-i\gamma^\mu)u^s(p)|_{k=p-p'} &= -i\bar{u}^{s'}(p')(\gamma^\mu)(p_\mu - p'_\mu)u^s(p) \\ = -i\bar{u}^{s'}(p')(\not{p}_\mu - \not{p}'_\mu)u^s(p) &= -i\bar{u}^{s'}(p')(m-m)u^s(p) = 0, \end{aligned} \tag{1.771}$$

by using the Dirac equation for both u and $\bar{u}$.

Having understood how the Ward identity makes a Lorentz invariant expression possible in cases like this we can now continue and square the amplitude for the unpolarised Compton scattering process. We get using the explicit form of the amplitude give above

$$|\mathcal{M}|^2_{unpol} = (\frac{1}{2})^2\Sigma_{s,s'}|\mathcal{M}|^2_{pol} = \frac{e^4}{4}g_{\mu\rho}g_{\nu\sigma}\times$$

$$tr\left((\not{p}'+m)(\frac{\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{\gamma^\nu\not{k}'\gamma^\mu - 2\gamma^\nu p^\mu}{2p\cdot k'})(\not{p}+m)(\frac{\gamma^\sigma\not{k}\gamma^\rho + 2\gamma^\rho p^\sigma}{2p\cdot k} + \frac{\gamma^\rho\not{k}'\gamma^\sigma - 2\gamma^\sigma p^\rho}{2p\cdot k'})\right). \tag{1.772}$$

There is a fair amount of gamma algebra here but if one writes out each non-zero term separately and use an identity we have discussed before, namely $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$ this algebra is not so bad. One finds at the end

$$|\mathcal{M}|^2_{unpol} = 2e^4\left(\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}) + 4m^2(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'})^2\right). \tag{1.773}$$

Compton scattering is typically an experiment where photons are hitting electrons at rest so we should get the kinematics in the *Lab* frame:

$$e^-: \;\; p^\mu = (m,0,0,0,), \;\; k^\mu = (\omega, \omega\hat{z}), \;\; p'^\mu = (E', \mathbf{p}'), \;\; k'^\mu = (\omega', \omega'\sin\theta, 0, \omega'\cos\theta). \tag{1.774}$$

Note that here the out-going particles (e^-, γ) can leave in any directions unrelated to each other. The kinematic relations are now easy to obtain:

$$p\cdot k = m\omega, \;\; p\cdot k' = m\omega'. \tag{1.775}$$

We also need to express ω' in terms of ω and the angle θ. This is done by rewriting m^2 as follows

$$m^2 = (p')^2 = (p+k-k')^2 = p^2 + 2p\cdot k - 2p\cdot k' - 2k\cdot k' = m^2 + 2m(\omega-\omega') - 2\omega\omega'(1+\cos\theta),$$

(1.776)

which is solved by Compton's formula

$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m}(1 - \cos\theta). \tag{1.777}$$

Using these results the following *Klein-Nishina formula* is a convenient way to express the final answer for the differential cross section

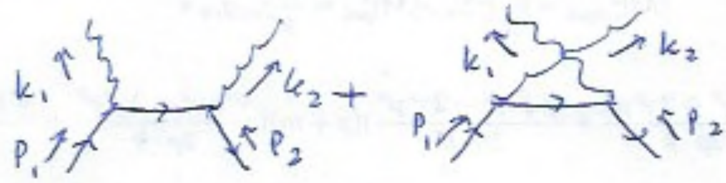
$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2 \left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right). \tag{1.778}$$

Here ω' is obtained from Compton's formula above.

Note that the low energy result found in experiments by Thomson is, for $\omega \to 0$ which implies $\frac{\omega'}{\omega} \to 1$,

$$\frac{d\sigma}{d\cos\theta} \to \frac{\pi\alpha^2}{m^2}(1 + \cos^2\theta), \;\; \sigma = \frac{8\pi\alpha^2}{3m^2}. \tag{1.779}$$

Pair annihilation into photons: This process is related to Compton scattering by crossing symmetry. Thus we can immediately write down the square of the scattering amplitude in this case. This process also requires two Feynman diagrams



Crossing symmetry with the substitution

$$p \to p_1, \;\; p' \to -p_2, \;\; k \to -k_1, \;\; k' \to k_2, \tag{1.780}$$

gives

$$|\mathcal{M}|^2_{unpol} = -2e^4\left(\frac{p_1\cdot k_2}{p_1\cdot k_1} + \frac{p_1\cdot k_1}{p_1\cdot k_2} + 2m^2\left(\frac{1}{p_1\cdot k_1} - \frac{1}{p_1\cdot k_2}\right) + 4m^2\left(\frac{1}{p_1\cdot k_1} - \frac{1}{p_1\cdot k_2}\right)^2\right). \tag{1.781}$$

The kinematics is again entirely different in the two crossing related cases so this has to be redone for the annihilation case: in the CM frame we have

$$p_1 = (E, p\hat{z}), \;\; p_2 = (E, -p\hat{z}), \;\; k_1 = (E, E\sin\theta, 0, E\cos\theta), \;\; k_2 = (E, -E\sin\theta, 0, -E\cos\theta). \tag{1.782}$$

This will give a differential cross section that in the high energy limit where the electron mass can be neglected gives

$$\frac{d\sigma}{d\cos\theta} \to \frac{2\pi\alpha^2}{s}\left(\frac{1+\cos^2\theta}{\sin^2\theta}\right). \tag{1.783}$$

Integrating over the final angle implies that one is over-counting since the two out-going photons are identical. Thus one has to divide the result by 2. This theoretical result is compared to experiment in a graph in PS, page 169.