1.13 Radiative corrections and renormalisation

1.13.1 Reading instructions for PS on Radiative corrections and renormalisation

Canvas: Lectures 15 - 20 will appear in one Doc-file (or maybe two) which will be updated with the latest lecture just before it starts. Also this page with reading instructions will be updated after each lecture to make it as accurate as possible.

We will study parts of chapters 6, 7 and 10 but jump back and forth in this material. The lectures 15 to 20 will define exactly what is important for this course. In particular these lectures will give the order in which to read this material. I advice you to follow this order at least the first time you go through it. It will be done in PS as follows:

1. Go back to Chap 1 and read pages 8 - 12 again, in particular the part called "Embellishment and Questions" and the comments connected to Figure 1.4.

2. Get more input on this QED discussion by reading Chap 6, Intro on p. 175 - 176.

3. To get into the subject of loop corrections and how to handle them when they are infinite we leave QED for now and turn to ϕ^4 theory which is much simpler than QED in this respect. Therefore we jump to Chap 10 where we read first the Intro, page 315, and turn to sect. 10.2 which we will study in all details (skipping sect. 10.1 for now). However, even sect. 10.2 will not be done in the order presented in PS: Instead we start from mid-page 326 "One-loop Structure of ϕ^4 theory" and return to the first part of sect. 10.2 after that. In section 10.2 the text refers back to chapter 7 a few times: the only thing needed from chapter 7 at this point is the stuff on "Dimensional regularisation" pages 249 - 251. You may also want to consult PS about Feynman parameters in Chap 6, pages end of 189 and 190, but the lectures will contain what you need.

4. Read then sect. 10.1, starting at eq. 10.12, pages 321 - 322: Counting divergencies in ϕ^4 theory.

5. Then study sections 10.1 and 10.3 (the rest of chapter 10 is not included).

6. read the Intro pages 175 - 176 again (sect. 6.1 is not included)
read sect 6.2 and 6.3 (sections 6.4 and 6.5 are not included)

7. Intro of Chap 7 and section 7.1. (Section 7.2 is not included)
Section 7.3: read pages 230 - 232 (the rest is not included)
Section 7.4: read page 238 (the rest is not included)
Section 7.5: This whole section is very important!

1.13.2 Renormalisation of ϕ^4 theory

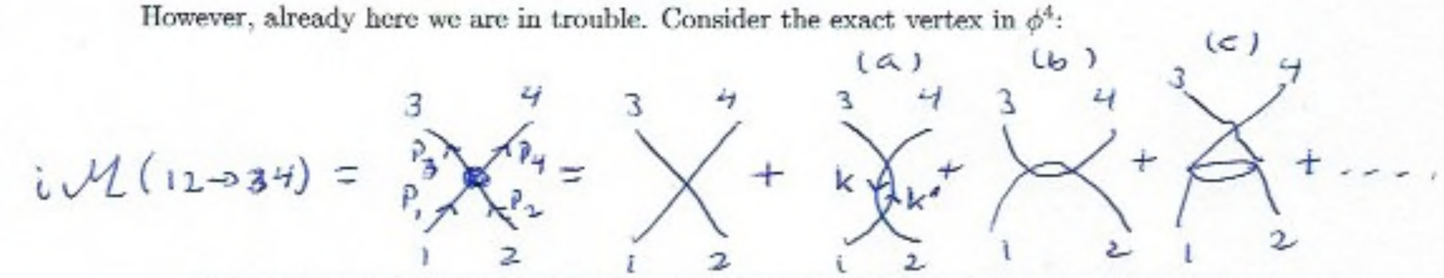
Renormalisation in QFT is a rather deep and tricky subject. Therefore it might be a good strategy to try to understand this first by looking at the simplest possible theory namely ϕ^4 theory. So consider again the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{1.784}$$

The whole issue of renormalisation concerns the following question:
What exactly do the parameters m and λ (and as we will see later also the field, here just ϕ) actually mean when the Lagrangian is used to make predictions for an experiment?

Naively we would just say that first we determine m and λ by performing two experiments which measure these parameters and then we can start predicting things by computing Feynman graphs and apply perturbation theory.

However, already here we are in trouble. Consider the exact vertex in ϕ^4:



To compare what we compute in perturbation theory (above) to the value of λ obtained in an experiment we see that the λ that appears in $\mathcal{L}$ is itself not the experimental value! This is clear since the tree-graph is given by the Feynman rule $-i\lambda$ but this is corrected by all the other terms in the above infinite series of terms. To solve this problem we will below introduce different λ parameters. But first we need to compute the first loop corrections drawn above to get a better feeling for the structure of the problem.

The series of terms above are given by the Feynman rules: $(k' = k + p_1 + p_2)$

$$i\mathcal{M}(p_1, p_2 \to p_3, p_4) = -i\lambda + (a) + (b) + (c) + ... \tag{1.785}$$

where

$$(a) = \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(k + p_1 + p_2)^2 - m^2 + i\epsilon}. \tag{1.786}$$

Here the factor $\frac{1}{2}$ comes from the symmetry factor $s = 2$ (two identical lines in the loop).

Define now $V(p^2)$ by

$$(a) := (-i\lambda)^2 iV(p^2)|_{p=p_1+p_2}. \tag{1.787}$$

This momentum is actually related to the Mandelstam variable $s = p^2 = (p_1 + p_2)^2$ and then we see that the expressions for the diagrams (b) and (c) are obtained by replacing s by the other Mandelstam variables $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$, respectively. The minus signs appear since p_3 and p_4 are out-going momenta.

Our next task is to analyse $V(p^2)$, that is, the four-dimensional momentum integral

$$V(p^2) := \frac{i}{2}\int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 - m^2 + i\epsilon}\,\frac{1}{(k + p_1 + p_2)^2 - m^2 + i\epsilon}. \tag{1.788}$$

This analysis is hard to do directly in the above integral so a very useful trick is to Wick rotate! That is, we can let k^0 become complex and then turn the integral over the real part of k^0 (appearing in the integral above) into an integral along the imaginary k^0 axis by letting $k^0 \to ik^0$. This is called a **Wick rotation** and can be viewed as a rotation of the real axis into the imaginary one without passing any poles in the Feynman propagators.

The integral has now become Euclidean, with k^2 replaced by $-k_E^2$, and we can therefore introduce polar coordinates in this four-dimensional Euclidean momentum space by

$$\int_{-\infty}^{\infty} \frac{d^4k_E}{(2\pi)^4} = \int_0^{\infty} dk k^3 \int_{S^3} d\Omega_4, \tag{1.789}$$

where k is now the radial coordinate in momentum space. The angular integral is over the unit radius 3-dimensional sphere S^3.

It is then a simple matter to check how $V(p^2)$ behaves for large momenta, i.e., in the UV limit. Introduce a large cut-off Λ in momentum space as follows

$$V(p^2) \propto \int^{\Lambda} dk\, k^3 \frac{1}{k^2}\frac{1}{k^2}, \tag{1.790}$$

where we have taken Λ big enough so that masses and external momenta p_i can be neglected in the denominator of $V(p^2)$. Then

$$V(p^2) \propto \int^{\Lambda} dk\, k^3 \frac{1}{k^2}\frac{1}{k^2} \propto \int^{\Lambda} \frac{dk}{k} \propto \log\Lambda \to \infty \text{ as } \Lambda \to \infty, \tag{1.791}$$

This is our first important result: the integral in $V(p^2)$ is divergent!

The above discussion and divergence analysis of $V(p^2)$ will force us to perform the following three steps:

1. Regularisation: This refers to the introduction of any kind of parameter (like Λ above) that can be used to define the way the integral approaches infinity when the parameter is taken to infinity (or to zero if that is how the divergence is emerging).

2. Renormalisation: This refers to the procedure required to relate the parameters in the Lagrangian to the measured values of these parameters. This step will also involve the fields themselves. The renormalisation needed here is multiplicative as we will see later.

3. Renormalisability: When the previous step is under control one can start checking if the theory is renormalisable. This is done by counting the divergent diagrams and comparing that number to the number of parameters in the Lagrangian (including the fields).

1. Regularisation: There are several ways to make a divergent integral convergent and physics must of course be independent of which one we use. This will be clear below. Here we will discuss three often used regularisations:
a) **Cut-off**: This is the one used above, that is after Wick rotation one introduces the cut-off parameter Λ by

$$\int_{-\infty}^{\infty} d^4k_E \to \int^{\Lambda} dk\, k^3 \int d\Omega, \tag{1.792}$$

which cuts of the integral at some large momentum Λ which is taken to infinity at the end. Note that this is an $SO(4)$ invariant procedure (corresponding to Lorentz invariance before Wick rotation) but it destroys gauge invariance in QED since in momentum space a gauge transformation reads $\delta A_\mu = ip_\mu\alpha$ and hence depends on the momentum. Thus it also affects unitarity.

b) **Pauli-Villars**: Here one introduces a heavy ghost particle of the same spin as the one in the divergent loop and then takes the mass M to infinity. Explicitly

$$D_F = \frac{i}{k^2 - m^2 + i\epsilon} \to \frac{i}{k^2 - m^2 + i\epsilon} - \frac{i}{k^2 - M^2 + i\epsilon}, \tag{1.793}$$

where the minus sign is the origin of the name *ghost* and heavy refers to the large value of M. This trick is Lorentz invariant and gauge invariant if applied to a photon propagator. It is not unitary until after the limit $M \to \infty$ is taken. The key point here is that for very large momenta where the masses can be neglected the two integrals cancel each other making the sum of the two terms UV finite. Of course, if $M \to \infty$ is taken first the second term is zero and we are back to the usual propagator.

c) **Dimensional regularisation**: Very nice to work with but physically a bit obscure perhaps. Here one generalises the momentum integrals to a general dimension d which does not even have to be integer:

$$d = 4 \to d = 4 - \epsilon \text{ where } \epsilon \to 0. \tag{1.794}$$

d) **Lattice regularisation**: Turning spacetime into a lattice makes it possible to compute certain quantities exactly. However, then the lattice must be removed by letting the lattice spacing go to zero which does not always work. Certain kind of chiral theories are also impossible to treat with this method. This will not be discussed any further in this course (see PS sect. 22.1).

The regularisation procedure raises a number of questions and it is therefore interesting to note that there are theories in four space-time dimensions which do not need this

step at all:
Field theory: Super-Yang-Mills with maximal number of supersymmetries known as $\mathcal{N} = 4$ SYM[41]. After this theory was discovered other less supersymmetric YM-theories have been found that are also finite.
String theory: Here there are no UV infinite diagrams at all at any loop order.

We now return to the integral $V(p^2)$ defined above which we found to be infinite. In order to compute it exactly (before Wick rotation) we need another trick due to Feynman: Introduce a **Feynman parameter** x by the following integral

$$\frac{1}{AB} = \int_0^1 dx \, \frac{1}{(xA + (1-x)B)^2}. \tag{1.795}$$

This is easily checked:

$$RHS = \left(-\frac{1}{xA + (1-x)B} \times \frac{1}{A-B}\right)_{x=0}^{x=1} = -\frac{1}{A-B}\left(\frac{1}{A} - \frac{1}{B}\right) = -\frac{1}{A-B} \times \frac{B-A}{AB} = \frac{1}{AB}. \tag{1.796}$$

Introducing this Feynman parameter into $V(p^2)$ while identifying A with $k^2 - m^2$ and B with $(k+p)^2 - m^2$ we get (the $+i\epsilon$ is not relevant here)

$$V(p^2) = \frac{i}{2} \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{(x(k^2-m^2) + (1-x)((k+p)^2 - m^2))^2}. \tag{1.797}$$

The expression in the denominator, often denoted D, can now be simplified somewhat (compare to PS p. 327):

$$D = x(k^2 - m^2) + (1-x)((k+p)^2 - m^2) = k^2 - m^2 + (1-x)(p^2 + 2k \cdot p). \tag{1.798}$$

Now we change integration variables from k to $l = k + (1-x)p$. This gives, still in Minkowski,

$$V(p^2) = \frac{i}{2} \int_0^1 dx \int \frac{d^4l}{(2\pi)^4} \frac{1}{(l^2 - \Delta)^2}, \text{ where } \Delta = m^2 - x(1-x)p^2. \tag{1.799}$$

This is a nice result since
1) V depends only p^2, not linearly on p^μ as before,
2) the l-integrand is independent of angles $\Rightarrow$ the Euclidean version of the integral is rather easy to compute exactly.

To compute $V(p^2)$ exactly we first Wick rotate: $l^0 := i\, l_E^0$ which implies

$$l^0 := i\, l_E^0 \Rightarrow l^2 = -l_E^2, \quad d^4l = i\, d^4 l_E. \tag{1.800}$$

[41] This theory was proven finite to all loop orders using superspace Feynman diagrams in the following papers (in chronological order) L. Brink, O. Lindgren and B.E.W. Nilsson, Nucl. Phys B212 (1983), S. Mandelstam, Nucl. Phys B213 (1983) and L. Brink, O. Lindgren and B.E.W. Nilsson, Phys. Lett. B123 (1983).

Therefore the Euclidean version of $V(p^2)$ is

$$V(p^2)_E = -\frac{1}{2}\int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{1}{(l_E^2+\Delta)^2}, \text{ where } \Delta = m^2 - x(1-x)p^2. \tag{1.801}$$

Here we should note that p^μ is still in Minkowski space although we have Wick rotated in the integration variable l^0 to be able to perform the integral as swiftly as possible.

The next step is therefore to do the angular integrals. Here we will take another important step and get the result in any dimension d and then let d be any real positive number. How this is possible will become clear below.

First we split the whole Euclidean momentum integral into a radial part and an angular part by

$$\int d^d l_E = \int dl\, l^{d-1} \int_{S^{d-1}} d\Omega_d, \tag{1.802}$$

which is just a direct generalisation of the cases $d=2$ and $d=3$. The radial coordinate in Euclidean momentum space is denoted l (without any index E) and S^{d-1} is the $d-1$-dimensional unit sphere.

The angular part can be done as follows. Recall that $\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}$ and hence, for integer values of d,

$$\pi^{\frac{d}{2}} = (\sqrt{\pi})^d = \int d^d x\, e^{-x_1^2-x_2^2-....-x_d^2} = \int_0^\infty dr\, r^{d-1} e^{-r^2} \int d\Omega_d. \tag{1.803}$$

But here we integral over the radial coordinate r is rather easily done by setting $y = r^2$. Then

$$\int_0^\infty dr\, r^{d-1} e^{-r^2} = \frac{1}{2}\int_0^\infty dy\, y^{\frac{d}{2}-1} e^{-y}. \tag{1.804}$$

This integral is quite remarkable since for $d=2$ it becomes

$$\int_0^\infty dy\, e^{-y} = [-e^{-y}]_0^\infty = 1. \tag{1.805}$$

Denoting the integral $\Gamma(d/2)$ for now we have $\Gamma(1) = 1$. By partial integrations one can prove that

$$\Gamma(n+1) = n\Gamma(n). \tag{1.806}$$

This is recursive and can be expressed as

$$\Gamma(n) = (n-1)!, \tag{1.807}$$

so this function is the standard *Gamma* function, here represented by the integral above.

However, the integral representation is not restricted to integer values of the argument n so we can define the angular integral above for any dimension d even when d is not an integer. Thus, for any real d, we have

$$\int d\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}. \tag{1.808}$$

To check this result, recall that $\Gamma(1/2)=\sqrt{\pi}$ and thus $\Gamma(3/2)=\frac{1}{2}\sqrt{\pi}$:

$$S^1: \int d\Omega_2 = \frac{2\pi^{\frac{2}{2}}}{\Gamma(\frac{2}{2})} = 2\pi, \;\; S^2: \int d\Omega_3 = \frac{2\pi^{\frac{3}{2}}}{\Gamma(\frac{3}{2})} = \frac{2\pi^{\frac{3}{2}}}{\frac{1}{2}\pi^{\frac{1}{2}}} = 4\pi. \tag{1.809}$$

For the case we are interested in here, that is $d=4$, we get

$$S^3: \int d\Omega_4 = \frac{2\pi^{\frac{4}{2}}}{\Gamma(\frac{4}{2})} = \frac{2\pi^2}{\Gamma(2)} = 2\pi^2. \tag{1.810}$$

Finally, we can summarise these results in the formula

$$\int d^d l_E = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \int_0^\infty dl\, l^{d-1}. \tag{1.811}$$

1. Regularisation: Now we can compute the integrals that appear in these loop corrections after Wick rotation:

$$I_l = \int_0^\infty dl\, l^{d-1} \frac{1}{(l^2+\Delta)^2} = (\text{set}\, y = l^2) = \frac{1}{2}\int_0^\infty dy\, y^{\frac{d}{2}-1} \frac{1}{(y+\Delta)^2}. \tag{1.812}$$

Set now

$$x = \frac{\Delta}{y+\Delta} \Rightarrow dx = -\frac{\Delta\, dy}{(y+\Delta)^2} \Rightarrow \frac{dy}{(y+\Delta)^2} = -\frac{dx}{\Delta}, \tag{1.813}$$

and solving for y we get

$$x = \frac{\Delta}{y+\Delta} \Rightarrow y = \frac{\Delta}{x} - \Delta = \Delta(\frac{1}{x}-1) = \Delta\frac{1-x}{x}. \tag{1.814}$$

Using these relations to turn the integral into an x-integral we get

$$I_l = \frac{1}{2}\int_0^1 \frac{dx}{\Delta}\, \Delta^{\frac{d}{2}-1} x^{1-\frac{d}{2}} (1-x)^{\frac{d}{2}-1} = \frac{1}{2}\Delta^{\frac{d}{2}-2} \int_0^1 dx\, x^{1-\frac{d}{2}} (1-x)^{\frac{d}{2}-1}. \tag{1.815}$$

In the spirit of the integral representation of the Γ function above one can now also express this integral in terms of Γ functions. The relation is provided by the following definition of the *Beta* function $B(\alpha,\beta)$:

$$B(\alpha,\beta) := \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{1.816}$$

Thus we have

$$I_l = \frac{1}{2}\Delta^{\frac{d}{2}-2} B(2-\tfrac{d}{2},\tfrac{d}{2}) = \frac{1}{2}\Delta^{\frac{d}{2}-2}\frac{\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(2)}. \tag{1.817}$$

This is a very interesting result: it is divergent for $d=4$ due to $\Gamma(2-\frac{d}{2}) = \Gamma(0) = \infty$. In fact, by analytic continuation one can show that $\Gamma(x)$ is finite for all real values of x except at non-positive integer values. To define this divergence in the physics problem we are looking at here, we let the dimension d become slightly less than 4, i.e. instead of using $d=4$ we insert $d=4-\epsilon$. This gives

$$\Gamma(2-\tfrac{d}{2})=\Gamma(\frac{\epsilon}{2})\approx\frac{2}{\epsilon}+\gamma+\mathcal{O}(\epsilon), \tag{1.818}$$

where we in the last step used a well-known expansion of the Γ function for small arguments. The constant γ is the Euler-Mascheroni constant $\gamma\approx 0.5772...$

At this point we should return to the question how physics can be extracted from these formulas. To do this we go back to the momentum integral in $V(p^2)$ which now is in d dimensions, and in the limit $\epsilon\to 0$ becomes

$$\int\frac{d^d l_E}{(2\pi)^d}\frac{1}{(l^2+\Delta)^2}|_{d=4-\epsilon}=\frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon}-\log\Delta-\gamma+\mathcal{O}(\epsilon)\right). \tag{1.819}$$

There are a couple of steps before one finds this result. The derivation goes as follows:

$$\int\frac{d^d l_E}{(2\pi)^d}\frac{1}{(l^2+\Delta)^2}=\int\frac{d\Omega_d}{(2\pi)^d}\int_0^\infty dl\, l^{d-1}\frac{1}{(l^2+\Delta)^2}=\frac{1}{(2\pi)^d}\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\frac{1}{2}\Delta^{\frac{d}{2}-2}\frac{\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(2)}. \tag{1.820}$$

This can be simplified a bit to, using also $\Gamma(2)=1$,

$$\int\frac{d^d l_E}{(2\pi)^d}\frac{1}{(l^2+\Delta)^2}=\frac{\pi^{\frac{d}{2}}}{(2\pi)^d}\Delta^{\frac{d}{2}-2}\Gamma(2-\tfrac{d}{2}). \tag{1.821}$$

To get the result quoted above we need not only the expansion of the last Γ factor given above, but also to expand $\Delta^{\frac{d}{2}-2}$ for small ϵ: with $d=4-\epsilon$ we get

$$\Delta^{\frac{d}{2}-2}=\Delta^{-\frac{\epsilon}{2}}=e^{-\frac{\epsilon}{2}\log\Delta}=1-\tfrac{\epsilon}{2}\log\Delta+\mathcal{O}(\epsilon^2). \tag{1.822}$$

Multiplying these two expansions together and collecting the $1/\epsilon$ and the ϵ independent terms gives the result above. Note that also the 2π factors could have been expanded like this and then contributed to the finite constant (momentum independent) terms. As will be clear later such terms contain no physics information which is why we skipped those terms here.

In fact, this is the point where we should stop and clarify where the physics information come from in the above expression

$$\int\frac{d^d l_E}{(2\pi)^d}\frac{1}{(l^2+\Delta)^2}|_{d=4-\epsilon}=\frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon}-\log\Delta-\gamma+\mathcal{O}(\epsilon)\right). \tag{1.823}$$

There are three kinds of terms here that survive the limit $\epsilon\to 0$: The first divergent one, the second finite but p^2 dependent one, and the third one which is just a finite constant. The renormalisation procedure to be discussed later when we fully understand the regularisation step discussed here will show that all information about the physics is contained in the $\log\Delta(p^2)$ term basically because of its p^2 dependence.

Having stated this fact we can continue to check if the other regularisation procedures generate the same physics. So let us complete the calculation with the cut-off parameter Λ. Then we need to compute, for $d = 4$,

$$I_l(\Lambda) = \int_0^\Lambda dl\, l^{d-1} \frac{1}{(l^2+\Delta)^2} = (y = l^2) = \frac{1}{2}\int_0^{\Lambda^2} dy\, y^{\frac{d}{2}-1} \frac{1}{(y+\Delta)^2}$$

$$= (d=4) = \frac{1}{2}\int_0^{\Lambda^2} \frac{dy\, y}{(y+\Delta)^2} = \frac{1}{2}\int_0^{\Lambda^2} dy\, y \partial_y (-\frac{1}{y+\Delta}) = \frac{1}{2}\int_0^{\Lambda^2} dy \frac{1}{y+\Delta} + \frac{1}{2}\left(-\frac{y}{y+\Delta}\right)\Big|_0^{\Lambda^2}$$

$$= \frac{1}{2}\log\frac{\Lambda^2+\Delta}{\Delta} - \frac{1}{2}\frac{\Lambda^2}{\Lambda^2+\Delta}. \tag{1.824}$$

In the limit $\Lambda \to \infty$ this reduces to

$$I_l(\Lambda\to\infty) = \log\Lambda - \frac{1}{2}\log\Delta - \frac{1}{2}, \tag{1.825}$$

and hence

$$\int \frac{d^d l_E}{(2\pi)^d}\frac{1}{(l^2+\Delta)^2}|_{\Lambda\to\infty} = \frac{1}{(4\pi)^2}(-\log\Delta + 2\log\Lambda - 1). \tag{1.826}$$

Recalling the rule stated above about which term contains the physics information, namely the one depending on p^2, i.e. $\log\Delta(p^2)$, we find that the physics is the same as for dimensional regularisation.

As a last case we also do the computation with Pauli-Villars regularisation. Thus, with the mass dependence in $\Delta(m) = m^2 - x(1-x)p^2$, we have

$$I_l(M) = \int_0^\infty dl\, l^3 \left(\frac{1}{(l^2+\Delta(m))^2} - \frac{1}{(l^2+\Delta(M))^2}\right) \tag{1.827}$$

which by setting $y = l^2$ becomes

$$= \frac{1}{2}\int_0^\infty dy\, y \left(\frac{1}{(y+\Delta(m))^2} - \frac{1}{(y+\Delta(M))^2}\right)$$

$$= \frac{1}{2}\int_0^\infty dy \left(\frac{1}{y+\Delta(m)} - \frac{1}{y+\Delta(M)}\right) - \frac{1}{2}\left(\frac{y}{y+\Delta(m)} - \frac{y}{y-\Delta(M)}\right)\Big|_0^\infty$$

$$= -\frac{1}{2}\log\frac{\Delta(m)}{\Delta(M)} = -\frac{1}{2}\log\Delta(m) + \frac{1}{2}\log\Delta(M). \tag{1.828}$$

Thus

$$\int \frac{d^d l_E}{(2\pi)^d}\frac{1}{(l^2+\Delta)^2}|_{M\to\infty} = \frac{1}{(4\pi)^2}(-\log\Delta(m) + \Delta(M)), \tag{1.829}$$

which once again provides the same physics information stored in the term $-\frac{1}{(4\pi)^2}\log\Delta(m)$ while the divergence is captured by the other term $\frac{1}{(4\pi)^2}\log\Delta(M)$.

We now turn to the issue of renormalisation which will explain the above statement about where to find the physics information after regularisation.

2. Renormalisation:

1.1.2 Renormalisation in ϕ^4 theory

Consider again the Lagrangian for ϕ^4 theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{1.47}$$

As already mentioned the whole issue of renormalisation concerns the following question:

Question: What exactly do the parameters m and λ actually mean when the Lagrangian is used to make predictions for an experiment? As we will see later this question also involves the field, here just ϕ.

Idea: Choose the parameters in $\mathcal{L}$ so that the observable quantities take their physical (finite) values.

How is this done? Considered again the scattering process discussed above

$$i\mathcal{M}(12 \to 34) = -i\lambda + (-i\lambda)^2(iV(s) + iV(t) + iV(u)) + ... \tag{1.48}$$

where, with $\Delta = m^2 - x(1-x)p^2$,

$$V(p^2) = -\frac{1}{32\pi^2}\int_0^1 dx\left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log(m^2 - x(1-x)p^2)\right) \tag{1.49}$$

and the dots indicate an infinite series of higher loop terms (at higher and higher order in the coupling constant λ).

Note now the following facts:

1) λ is not the physical coupling constant λ_{phys} measured in experiments since that value is determined by the whole perturbation series, i.e.,

$$i\mathcal{M} = -i\lambda_{phys}(p^2), \tag{1.50}$$

which does depend on p^2, the momentum at which the scattering experiment is performed. This p^2 dependence is seen in experiments so it is not surprising that also the theory, via $V(p^2)$, indicates that p^2 plays a role here. We will state this fact as

$$\lambda_{measured} = \lambda_{phys}(p^2). \tag{1.51}$$

We emphasise here that the Lagrangian $\mathcal{L}(x)$ can NOT contain parameters that depend on momenta since that would make it non-local or worse.

2) Our intuition that coupling constants are constant come from experiments at very low energies and therefore it is natural to define "the coupling constant" λ at zero 3-momentum $\mathbf{p} = 0$, called the **subtraction point**, by

$$\lambda := \lambda_{phys}(p^\mu = (m, 0, 0, 0)). \tag{1.52}$$

This fixed number λ determined by experiment is the value we will use at the end in the so called **renormalised Lagrangian**.

The standard notation used in the QFT literature is to denote quantities like the mass and the coupling constant appearing in the Lagrangian as λ_0 and m_0, called as the **bare** quantities since λ and m will, as just noted above, refer to the finite physical (i.e., measured) values at the subtraction point. This is just a renaming of the parameters in $\mathcal{L}$!

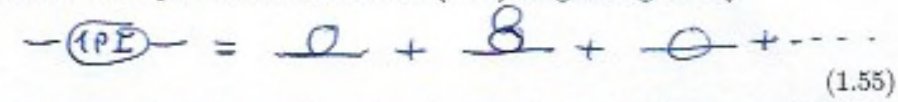
So let us restart the discussion this time from the Lagrangian with the bare parameters

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4. \tag{1.53}$$

Now we need to discuss also the role of the field ϕ and at the same time the mass m_0. To do this we consider the exact two-point function, i.e., the propagator, in perturbation theory:

$$\begin{aligned} &= \frac{i}{p^2 - m_0^2 + i\epsilon} + \frac{i}{p^2 - m_0^2 + i\epsilon}(-iM(p^2))\frac{i}{p^2 - m_0^2 + i\epsilon} \\ &+ \frac{i}{p^2 - m_0^2 + i\epsilon}(-iM(p^2))\frac{i}{p^2 - m_0^2 + i\epsilon}(-iM(p^2))\frac{i}{p^2 - m_0^2 + i\epsilon} + \ldots \end{aligned} \tag{1.54}$$

where we have defined the **1-particle irreducible (1PI)** diagrams given by

$-iM(p^2) =$ 

(1.55)

Then the exact propagator is a geometric series that can be summed up, provided the $M(p^2)$ is small enough, to give

$$\frac{i}{p^2 - m_0^2}\left(\frac{1}{1 - \frac{M(p^2)}{p^2 - m_0^2}}\right) = \frac{i}{p^2 - m_0^2 - M(p^2) + i\epsilon}. \tag{1.56}$$

As we have seen already the quantity $M(p^2)$ will be infinite if not regularised (but always close to infinite), and so will m_0 but in a way that cancels in the combination in the propagator $m_0^2 - M$. This is not an acceptable situation (e.g., in summing up the geometric series). This problem is eliminated in the renormalised Lagrangian that will be defined below!

This form of the exact propagator has two important implications:

1) The physical mass of the particle is determined by the value of p^2 where the propagator diverges, that is for the p^2 that solves the equation

$$p^2 - m_0^2 - M(p^2) = 0. \tag{1.57}$$

Call this value m^2, that is by definition $p^2 = m^2$. This is the mass we obtain from an experiment and this value of p^2 plays the role of subtraction point in this case.

2) Let us now expand $M(p^2)$ around this physical value m^2:

$$M(p^2) = M(m^2) + (p^2 - m^2)\left(\frac{d}{dp^2}M(p^2)\right)|_{m^2} + ... \tag{1.58}$$

Inserting this expansion into the propagator above gives

$$\frac{i}{p^2 - m_0^2 - M(p^2) + i\epsilon} = \frac{i}{p^2 - m_0^2 - (M(m^2) + (p^2 - m^2)(\frac{d}{dp^2}M(p^2))|_{m^2} + ...) + i\epsilon}. \tag{1.59}$$

This means that for p^2 close to the pole $p^2 - m_0^2 - M(m^2) = p^2 - m^2$ and the propagator reads

$$\frac{i}{(p^2 - m^2)(1 - \frac{d}{dp^2}M(p^2))|_{m^2}} := \frac{iZ}{p^2 - m^2}, \tag{1.60}$$

where we can identify the **field renormalisation constant** Z as

$$Z := \left(1 - \frac{d}{dp^2}M(p^2)|_{m^2}\right)^{-1}. \tag{1.61}$$

Since the exact propagator is really the two-point function $\langle\Omega|T\phi^{exact}(x)\phi^{exact}(y)|\Omega\rangle$ we want it to behave as $\frac{i}{p^2-m^2+i\epsilon}$ close the pole where m is the physical mass. It is therefore convenient to define a new field ϕ_r, the **renormalised** field, by rescaling the field in the Lagrangian ϕ as follows

$$\phi = \sqrt{Z}\phi_r. \tag{1.62}$$

The Lagrangian is then written as

$$\mathcal{L} = \frac{1}{2}Z\partial_\mu\phi_r\partial^\mu\phi_r - \frac{1}{2}Z\, m_0^2\phi_r^2 - \frac{\lambda_0}{4!}\, Z^2\, \phi_r^4, \tag{1.63}$$

and now the key point about this version of the Lagrangian is that the exact propagator close to the pole is precisely

$$\langle\Omega|T\phi_r(x)\phi_r(y)|\Omega\rangle = \frac{i}{p^2 - m^2 + i\epsilon} + ... \tag{1.64}$$

Now one could start doing perturbation theory using the standard Feynman rules but expressed in terms of the new renormalised field ϕ_r and the bare constants m_0 and λ_0. However, there is a much more convenient way to view this Lagrangian which emerges if one defines the following δ-parameters:

$$\delta_Z := Z - 1,\ \ \delta_m := m_0^2 Z - m^2,\ \ \delta_\lambda := \lambda_0 Z^2 - \lambda. \tag{1.65}$$

Expressing the Lagrangian in terms of these δ parameters and the physical quantities m, λ and the field ϕ_r, instead of the bare parameters m_0 and λ_0 together with ϕ has no physical effect but gives the renormalisation procedure a very appealing structure as we will see. Remember that λ and m are just fixed numbers so these equations are just replacing Z, λ_0, m_0 by their respective δ parameters $\delta_Z, \delta_\lambda, \delta_m$.

This change of parameters gives the Lagrangian the following form:

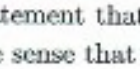
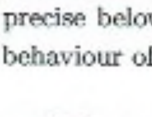
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_r\partial^\mu\phi_r - \frac{1}{2}m^2\phi_r^2 - \frac{\lambda}{4!}\phi_r^4 \tag{1.66}$$

$$+ \frac{1}{2}\delta_Z\partial_\mu\phi_r\partial^\mu\phi_r - \frac{1}{2}\delta_m m^2\phi_r^2 - \frac{1}{4!}\delta_\lambda\phi_r^4. \tag{1.67}$$

The fundamental interpretation of this **renormalised** Lagrangian is the following:

1) The first line of terms generate the standard Feynman rules and look exactly the same as they did in the beginning but the parameters are now the physical measured ones, a statement that will be made more precise below. Also the field ϕ^r is the physical one in the sense that it gives the physical behaviour of the exact propagator close to the pole.

2) The second line contains the so called **counter terms** and they give rise to a new set of Feynman rules:

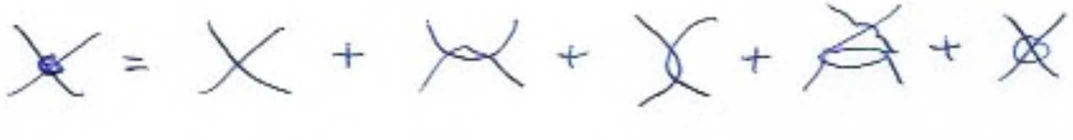
 $= i(\delta_Z\, p^2 - \delta_m),$  $= -i\delta_\lambda.$ (1.68)

Note that the first term on the second line in $\mathcal{L}$ looks like a kinetic term but that we are now interpreting it as an interaction term!

The huge advantage of this renormalised Lagrangian over any other equivalent form will be obvious when we start using it. However, we still need to be precise about what the physical parameters m and λ in $\mathcal{L}$ are. The discussion in the beginning about the physical coupling constant suggest what to do. We should, quite arbitrarily, pick a momentum, called the **subtraction point(s)**, at which we decide the measured value to be the one that is given to the m and λ appearing in $\mathcal{L}$. Such a choice is necessary for at least two reasons:

1) The measured values of these parameters depend on p^2,
2) The parameters in the Lagrangian must be independent of p^2.

Let us now apply these ideas in a couple of explicit cases to see how it works. First we return to the scattering $12 \to 34$ in ϕ^4 theory. We saw above that the tree diagram is corrected at one-loop level by three diagrams represented by a function $V(p^2)$ where $p^2 = s, t, u$, respectively, for the three different channels. In renormalised ϕ^4 theory there is one more diagram coming from the new interaction vertex $-i\delta_\lambda$. Thus we have now the diagram expansion



which gives the scattering amplitude

$$i\mathcal{M}(12 \to 34) = -i\lambda + (-i\lambda)^2(iV(s) + iV(t) + iV(u)) - i\delta_\lambda. \tag{1.69}$$

The first thing to do is to determine the constant (i.e., p^2 independent) value of δ_λ. This is done by demanding that at the subtraction point, which in case we choose to be $\mathbf{p_1} = \mathbf{p_2} = 0$, the counter term cancels the one-loop terms so that

$$\mathcal{M}(12 \to 34)|_{subt.point} = -i\lambda. \tag{1.70}$$

In other words: at the subtraction point the physically measured value of the coupling constant, i.e. $\mathcal{M}(12 \to 34)|_{subt.point}$, is the value we give the coupling constant, λ, in the Lagrangian.

With this choice of subtraction point we have $s = (p_1 + p_2)^2 = 4m^2$. Then since $s + t + u = 4m^2$ we also have that $t + u = 0$ which implies $E_3 + E_4 = 2m$, and thus also that $\mathbf{p_3} = \mathbf{p_4} = 0$. The subtraction point is therefore given by $s = 4m^2$ and $t = u = 0$. Thus

$$\delta_\lambda = -\lambda^2(V(4m^2) + 2V(0)). \tag{1.71}$$

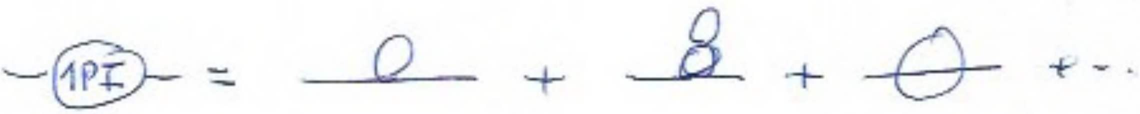
Using this result we can finally write down the scattering amplitude for any momenta p_1 and p_2:

$$i\mathcal{M} = -i\lambda - \frac{i\lambda^2}{32\pi^2}\int_0^1 dx \left(\log\frac{m^2 - x(1-x)s}{m^2 - x(1-x)4m^2} + \log\frac{m^2 - x(1-x)t}{m^2} + \log\frac{m^2 - x(1-x)u}{m^2}\right). \tag{1.72}$$

This formula explains all the subtle features of renormalised perturbation theory:
1) The Lagrangian is well-defined since the coupling constant λ appearing in it is a constant, whose value is exactly the one measured at the subtraction point: at the subtraction point the above equation becomes $i\mathcal{M} = -i\lambda$.
2) The measured value of the coupling constant at general momenta, λ_{phys}, is the value of $i\mathcal{M}(p^2)$ which can be computed in perturbation theory as done here to first loop-order.
3) The counter term is a sum of constant pieces, finite or infinite, at each power in λ such that they exactly cancel the corresponding terms that arise in the loop calculations. All kinds of regularisation parameters can then be eliminated (i.e., taken to infinity or zero) leaving only finite results.

Comment: In QED the analogues calculation can be done and compared to experiment and the renormalised theory is found to work extremely well. In other words, as suggested by renormalised QED the electric charge e is not a constant but depends on the energy scale at which the experiment is performed. This is also exactly what is seen in experiments: in terms of the fine structure constant which takes it usual value $1/137$ at low energy (or large scale) there is a 5 per cent increase in its value going from the subtraction point at low energy to 30 GeV. This fact will be given a quite intuitive explanation in the very last lecture.

Having understood how renormalised perturbation theory works for the coupling constant let us turn to the slightly more complicated case of the renormalisation of the mass m and the field ϕ. The theory now gives the $1PI$ blob as



which gives schematically without precise symmetry factors etc (this is M not $\mathcal{M}$)

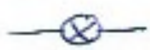
$$-iM(p^2) = -i\lambda\frac{1}{2}\int\frac{d^4k}{k^2} + (-i\lambda)^2\int\frac{d^4k_1d^4k_2}{k_1^2(k_2^2)^2} + (-i\lambda)^2\int\frac{d^4k_1d^4k_2}{k_1^2k_2^2(p-k_1-k_2)^2} + \dots \quad (1.73)$$

Let us first compute the snail-diagram at order λ using the standard methods. Wick rotation $k^0 \to ik^0$, $d^dk \to id^dk$ and $k^2 - m^2 \to -(k_E^2 + m^2)$, gives

$$-\frac{i\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2} \to -\frac{i\lambda}{2}\int\frac{d^dk_E}{(2\pi)^d}\frac{1}{k_E^2+m^2} = -\frac{i\lambda}{2(2\pi)^d}\int dk\,k^{d-1}\frac{1}{k^2+m^2}\int d\Omega_d \quad (1.74)$$

$$= -\frac{i\lambda}{2}\frac{(m^2)^{\frac{d}{2}-1}}{(4\pi)^{\frac{d}{2}}}\Gamma(1-\tfrac{d}{2}). \quad (1.75)$$

To this result for the snail graph we must now add the counter term



$$= i(p^2\delta_Z - \delta_m), \quad (1.76)$$

and demand that at the subtraction point, being $p^2 = m^2$ in this case (i.e., not the same as for the vertex),

$$M(p^2)|_{p^2=m^2} = 0, \quad \frac{d}{dp^2}M(p^2)|_{p^2=m^2} = 0, \quad (1.77)$$

which implies

$$\delta_Z = 0, \quad \delta_m = -\frac{\lambda}{2}\frac{(m^2)^{\frac{d}{2}-1}}{(4\pi)^{\frac{d}{2}}}\Gamma(1-\tfrac{d}{2}). \quad (1.78)$$

Since the snail graph is p^2 independent the end result is that the counter term cancels out the snail graph completely. Note that this renormalisation also involves the cancellation of an infinite piece coming from $\Gamma(1-\frac{d}{2})$. In fact, the snail graph is Λ^2 divergent which in dimensional regularisation translates to

$$\Gamma(1-\tfrac{d}{2})|_{d=4-\epsilon} = \Gamma(-1+\tfrac{\epsilon}{2}) = \frac{\Gamma(\frac{\epsilon}{2})}{-1+\frac{\epsilon}{2}}. \quad (1.79)$$

Expanding these two factors for small ϵ we get

$$\Gamma(1-\tfrac{d}{2})|_{d=4-\epsilon} = -\left(\tfrac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)\right)\left(1+\tfrac{\epsilon}{2}+\mathcal{O}(\epsilon^2)\right) = -\left(\tfrac{2}{\epsilon} - \gamma + 1 + \mathcal{O}(\epsilon)\right), \quad (1.80)$$

which is infinite but has no term relevant for physics and hence it gets completely cancelled by the counter term.

Thus the interesting points here are

1. The first (the snail) is infinite (goes as Λ^2) but independent of p^2.
2. The same is true for the second term (the double-snail).
3. The third term (the sunset) is also infinite (goes also as Λ^2) but does depend on p^2.

Let us investigate the third term a bit more. Let us call the function this diagram generates $f(p^2)$. The following argument is, in fact, applicable even if this function represents the entire series of perturbation terms. The integral in the third term is as we saw above

$$f(p^2) := (-i\lambda)^2 \int \frac{d^4k_1 d^4k_2}{k_1^2 k_2^2 (p-k_1-k_2)^2}. \tag{1.81}$$

We know from the Feynman parameter trick (together with a shift in the integration variables) that this integral can be made to depend only on p^2 and not linearly onp^μ. Let us then Taylor expand this function around the subtraction point $p^2 = m^2$:

$$f(p^2) = f(m^2) + p^2 \left(\frac{d}{dp^2} f(p^2) \right) |_{p^2=m^2} + \frac{1}{2}(p^2)^2 \left((\frac{d}{dp^2})^2 f(p^2) \right) |_{p^2=m^2} + ... \tag{1.82}$$

The interesting thing that happens here is that the degree of divergence of the integral decreases with each extra p^2 derivative:

$$\frac{d}{dp^2} \int \frac{d^2k_1 d^4k_2}{k_1^2 k_2^2 (p-k_1-k_2)^2} \propto \int \frac{d^2k_1 d^4k_2}{k_1^2 k_2^2 (p-k_1-k_2)^4}. \tag{1.83}$$

Thus the derivative turns the Λ^2 divergent integral on the LHS into the $\log \Lambda$ divergent integral on the RHS. Doing another derivative will therefore produce a convergent integral that goes as Λ^{-2} as $\Lambda \to \infty$. Thus the sunset graph gives rise to (as would also the entire series of terms) two infinite constants at order λ^2 that must be cancelled in the subtraction procedure. We express this as follows

$$f(p^2)_{snail}|_{div} \propto \Lambda^2 + p^2 \log \Lambda + finite. \tag{1.84}$$

Both divergent term will show up in dimensional regularisation as simple poles in ϵ and can thus be cancelled by adding new infinite terms at order λ^2 to the counter terms δ_m and δ_Z where the latter one does indeed multiply p^2 in the Lagrangian. This cancellation procedure can in principle be carried out to arbitrary order in perturbation theory and the *de*-parameters will therefore be infinite power series in the coupling constant λ.

The fundamental question that must now be asked is: What happens if there are infinite Feynman graphs appearing in the scattering process $2 \to 4$ which does not correspond to a term in the Lagrangian and hence cannot be cancelled against a counter term? The only way out of this dilemma is to add the corresponding interaction term to the Lagrangian so also these infinities can be cancelled by a counter term. Then the hope must be that it stops here or just continues a finite number of times. However, this is not the case: once one adds one single non-renormalisable term an infinite set of higher interaction terms must be added and the whole theory becomes non-renormalisable in the sense that an infinite number of experiments must be performed before a prediction can be made. Our next task is therefore to find a way to identify these dangerous terms that will render a theory useless in this sense.

Comment: This situation should be compared to what happens in gravity which is non-renormalisable but still a very useful theory! Recall that the Einstein-Hilbert Lagrangian $\mathcal{L}_{EH} = \sqrt{-\det g}R$ contains both the metric $g_{\mu\nu}$ and its inverse so if one expands it in terms of $h_{\mu\nu}$ defined by $g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{8\pi G}h_{\mu\nu}$ (using standard GR conventions with $g_{\mu\nu}$ the curved metric and $\eta_{\mu\nu}$ the flat Minkowski one) the Lagrangian becomes an infinite series of terms in powers of $h_{\mu\nu}$ all with two derivatives and the indices contracted in more and more complicated ways. However complicated this $\mathcal{L}_{EH}$ is in this expansion around Minkowski space its first term is just a conventionally normalised kinetic one $\frac{1}{2}(\partial_\mu h_{\nu\rho})^2$ while the next one is schematically $\sqrt{8\pi G}h\partial h\partial h$ etc for the following higher order terms. Thus the coupling constant in Einstein's general relativity is $\sqrt{8\pi G}$ which has dimension L^1 making the theory non-renormalisable in the sense defined above.

1.1.3 Renormalisability of ϕ^4 theory

In order to discuss the issue of whether a theory is renormalisable or not we must first investigate where, that is in which Feynman diagrams, divergencies occur. Then the question is whether all divergencies can be cancelled by good counter terms, that is counter terms that do not lead to even more divergencies. We will do tis in two steps:
1. Count divergencies.
2. Check renormalisability.

Counting divergencies in d-dimensional ϕ^n: Consider again the Lagrangian for the ϕ^4 but now generalised to any dimension d, i.e., d can be 2,3,4,5 or any other integer number, and with an order n interaction term ϕ^n (for some positive integer n):

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{n!}\phi^n. \tag{1.85}$$

For a general Feynman diagram we denote the number of the different parts as follows:

$$\begin{array}{ll} N & \text{number of external legs,} \\ V & \text{number of verticies,} \\ P & \text{number of propagators} \\ L & \text{number of loops.} \end{array} \tag{1.86}$$

Then we define the **superficial degree of divergence**, denoted D, by

$$D := dL - 2P, \tag{1.87}$$

since each loop contributes the dimension of d^dk (i.e., d) and each propagator the dimension of $1/k^2$ (i.e., -2) to the total dimensionality of the integral. This definition is exactly what we used above when discussing the dependence of loop integrals in ϕ^4 theory on the cut-off parameter Λ. If D turns out to be non-negative the integral diverges as Λ^D or $\log\Lambda$ if $D = 0$. The name "superficial" is used because there are complicated Feynman diagrams where D does not give the whole story. We will only discuss one such case, namely gauge theory in the form of QED.

First some examples from ϕ^3 theory in six dimensions:

$$= \int^\Lambda d^6k\,\frac{1}{k^2}\,\frac{1}{(k+p)^2} \propto \Lambda^2,\ \text{ OK since } D = 6\cdot 1 - 2\cdot 2 = 2. \tag{1.88}$$

$$= \int^\Lambda d^6k_1 \int^\Lambda d^6k_2\,(\frac{1}{k^2})^5 \propto \Lambda^2,\ \text{ OK since } D = 6\cdot 2 - 2\cdot 5 = 2. \tag{1.89}$$

$$= \int^\Lambda d^6k\,(\frac{1}{k^2})^3 \propto \log\Lambda,\ \text{ OK since } D = 6\cdot 1 - 2\cdot 3 = 0. \tag{1.90}$$

$$= \int^\Lambda d^6k\,(\frac{1}{k^2})^4 \propto \Lambda^2,\ \text{ finite, OK since } D = 6\cdot 1 - 2\cdot 4 = -2. \tag{1.91}$$

These examples are quite trivial but we can learn a lot more from them:
–D=2 is the maximal value.
–D decreases with increased number of external legs.
–Adding a new internal line (propagator) adds $6-2\cdot 3=0$ to D, i.e., the number of vertices does not matter so D is the same to all orders in perturbation theory (i.e., in powers of λ): it only depends on the number N of external legs of the diagram.

All these conclusions are also true for ϕ^4 theory in four dimensions but not if either d or n is changed in these cases (i.e., $d=6$ is tied to ϕ^3 and $d=4$ is tied to ϕ^4 for these conclusions to be true). It is rather easy to derive a general formula for D which contains these facts as special cases.

To do this we make use of two relations between the numbers defined above, N, V, P and L:

$$nV = N + 2P. \tag{1.92}$$

This is a direct consequence of the fact that each externa leg connects one end and each internal line connects two ends to the available vertices in the Feynman diagram.

$$L = P - V + 1. \tag{1.93}$$

This relation follows by counting the total number momentum integrals and momentum delta-functions δ^4 using x-space Feynman rules: The loop integrals all come from the propagators $\int d^d p e^{ip\cdot x}/p^2 - m^2$ but some of them can be done using the delta functions coming from the vertices $\int d^d x \Rightarrow \delta^d(momenta)$: Thus we have $L = P-(V-1)$ where the 1 refers to the δ^4 implementing overall momentum conservation that remains at the end.

Using these relations to eliminate first L and then P from D we find:

$$(\phi^n)|_d: \quad D = d + \left(n\frac{d-2}{2} - d\right)V - \frac{d-2}{2}N. \tag{1.94}$$

This is an extremely nice formula containing a lot of information:
1. Increasing $N \Rightarrow$ decreasing D for $d>2$ (as we saw above).
2. $d=2$ is very special: $D = 2-2V$ independent of N.
3. In d dimensions a scalar field has length dimension $[\phi] = L^{-\frac{d-2}{2}}$. Thus the coupling constant λ_n from the interaction term discussed here $\frac{\lambda_n}{n!}\phi^n$ has dimension $[\lambda_n] = L^{n\frac{d-2}{2}-d}$. Thus we see that the two special cases analysed above $(d,n)=(6,3)$ and $(d,n)=(4,4)$, both have dimensionless coupling constants since $d = n\frac{d-2}{2}$ in both cases. Therefore we have that

$$n = \frac{2d}{d-2} \Rightarrow D = d - \frac{d-2}{2}N, \tag{1.95}$$

and the conclusion we found above by looking at these special cases follow directly. There is a third quite interesting case of this kind, namely $(d,n)=(3,6)$[2]

[2]This case is relevant in M-theory.

Comment: In all these cases with a dimensionless coupling constant the massless theory has more space-time symmetry than Poincaré, they are invariant under the conformal group. The conformal group is the Poincaré plus scale transformations, which is the symmetry of the light-cone $ds^2 = g_{\mu\nu}dx^\mu x^\nu = 0$. Also Maxwell's theory is conformal as well as QED with massless fermions (since e is dimensionless), and, in fact, the whole of the standard model is conformal before the Higgs effect if we drop the mass term for the Higgs field.

Consider now a scalar theory i d dimensions with a ϕ^p interaction term where $p < n$. Then

$$D = d + \left(p\frac{d-2}{2} - d\right)V - \frac{d-2}{2}N, \text{ where } p < n = \frac{2d}{d-2}, \tag{1.96}$$

implies that the bracket is *negative* and hence D decreases with increasing number of vertices for any N, but if $p > n$ D will become positive (with new infinite diagrams appearing) for large enough V for any N.
The condition $n = \frac{2d}{d-2}$ on the power of the interaction term for the coupling constant to be dimensionless is thus a boarder case between ϕ^p with $p < n$ and $p > n$ that gives rise to the following classification of scalar field interactions and, in fact, theories in general:

Finite: Has no infinite Feynman diagrams at all.
Ex: String theory and $\mathcal{N} = 4$ SYM.

Superenormalisable: Finite number of infinite diagrams, $[\lambda] = L^{<0}$.
Ex: $[m^2] = L^{-2}$ (such interactions called *relevant* in condensed matter physics).

Renormalisable: Infinite number of infinite diagrams but only for small N, $[\lambda] = L^{\leq 0}$.
Ex: ϕ^4 in $d = 4$ (called *marginal* in condensed matter).

Non-renormalisable: Infinite number of infinite diagrams at all values of N, $[\lambda] = L^{>0}$.
Ex: Gravity with $[G] = L^2$ (called *irrelevant* in condensed matter.)

To make these ideas concrete let us return to ϕ^4 in $d = 4$ theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_r\partial^\mu\phi_r - \frac{1}{2}m^2\phi_r^2 - \frac{\lambda}{4!}\phi_r^4. \tag{1.97}$$

This theory has three quantities

$$[\phi] = L^{-1}, \quad [m] = L^{-1}, \quad [\lambda] = L^0, \tag{1.98}$$

that are associated with the renormalisation constants δ_Z, δ_m and δ_λ which can be used to cancel infinities arising in perturbation expansion. The number of infinities must therefore not exceed three (and should also be checked to appear in the right places to be cancelled):

$$D = 4 - N \Rightarrow \begin{cases} N = 2 : D = 2 \to \Lambda^2 \Rightarrow 2 \text{ infinities} \\ N = 4 : D = 0 \to \log\Lambda \Rightarrow 1 \text{ infinity} \\ N = 6 : D = -2 \to \Lambda^{-2} \Rightarrow \text{no infinities.} \end{cases} \tag{1.99}$$

The conclusion from this counting exercise is that the number of infinities is 3 which equals the number of renormalisation constants in the theory making it *renormalisable*.

Comment: As we will see later this counting of infinities and renormalisation constants becomes more involved but also much more interesting when symmetries enter the situation, either gauge symmetries as in QED of global ones as for instance $\phi \to -\phi$ which is present in the ϕ^4 in $d = 4$ theory discussed here.

To exemplify this comment we consider instead Yukawa theory with a Dirac fermion coupled via $g\phi\bar{\psi}\psi$ to a real scalar with self-interaction $\lambda_3\phi^3$ in $d = 4$. This theory is renormalisable according to the classification above since g is dimensionless and λ_3 has dimension L^{-1}. But the scalar potential is potentially bad since it is unbounded from below and the theory is therefore unstable. However, let us consider the perturbation theory despite this fact. There are a number of divergent graphs, the bosonic snail, the fermionic snail, the fermi triangle, the fermi square etc: (solid line=fermion, dashed line =scalar)

The superficial degree of divergence for these diagrams is: the fermionic triangle has $D = 1$ and the fermionic square has $D = 0$. Thus the theory is not renormalisable since $\mathcal{L}$ does not contain a ϕ^4 term which is generated in perturbation theory by a diagram that is infinite. The conclusion is that we have to add the ϕ^4 term to the Lagrangian which then becomes both renormalisable and stable.

Exercise: Verify that this Yukawa theory with both cubic and quartic scalar interaction terms is renormalisable by counting divergencies and renormalisation constants.

The above conclusion about the need to add the quartic term to the cubic one is very general and can be expressed as follows:

Rule for renormalisability: All possible renormalisable terms must be included in the Lagrangian unless they are forbidden by symmetries.

An example of this is the ϕ^4 in $d = 4$ theory which does not force us to add the cubic term. The reason being that the global symmetry $\phi \to -\phi$ makes it impossible for the theory to generate any non-zero three-point functions. We will encounter other examples of this phenomenon later.