1.13 Radiative corrections and renormalisation

1.13.1 Reading instructions for PS on Radiative corrections and renormalisation

Canvas: Lectures 15 - 20 will appear in one Doc-file (or maybe two) which will be updated with the latest lecture just before it starts. Also this page with reading instructions will be updated after each lecture to make it as accurate as possible.

We will study parts of chapters 6, 7 and 10 but jump back and forth in this material. The lectures 15 to 20 will define exactly what is important for this course. In particular these lectures will give the order in which to read this material. I advice you to follow this order at least the first time you go through it. It will be done in PS as follows:

1. Go back to Chap 1 and read pages 8 - 12 again, in particular the part called "Embellishment and Questions" and the comments connected to Figure 1.4.

2. Get more input on this QED discussion by reading Chap 6, Intro on p. 175 - 176.

3. To get into the subject of loop corrections and how to handle them when they are infinite we leave QED for now and turn to ϕ^4 theory which is much simpler than QED in this respect. Therefore we jump to Chap 10 where we read first the Intro, page 315, and turn to sect. 10.2 which we will study in all details (skipping sect. 10.1 for now). However, even sect. 10.2 will not be done in the order presented in PS: Instead we start from mid-page 326 "One-loop Structure of ϕ^4 theory" and return to the first part of sect. 10.2 after that. In section 10.2 the text refers back to chapter 7 a few times: the only thing needed from chapter 7 at this point is the stuff on "Dimensional regularisation" pages 249 - 251. You may also want to consult PS about Feynman parameters in Chap 6, pages end of 189 and 190, but the lectures will contain what you need.

4. Read then sect. 10.1, starting at eq. 10.12, pages 321 - 322: Counting divergencies in ϕ^4 theory.

5. Then study sections 10.1 and 10.3 (the rest of chapter 10 is not included).

6. read the Intro pages 175 - 176 again (sect. 6.1 is not included)
read sect 6.2 and 6.3 (sections 6.4 and 6.5 are not included)

7. Intro of Chap 7 and section 7.1. (Section 7.2 is not included)
Section 7.3: read pages 230 - 232 (the rest is not included)
Section 7.4: read page 238 (the rest is not included)
Section 7.5: This whole section is very important!

1.13.2 Renormalisation of ϕ^4 theory

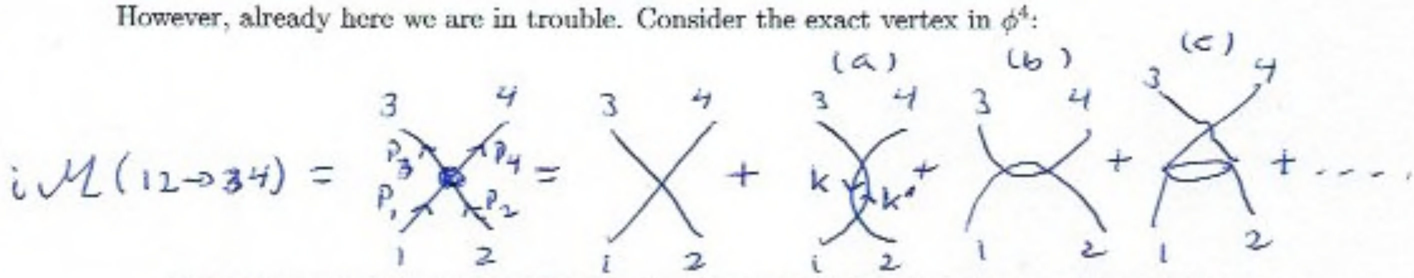
Renormalisation in QFT is a rather deep and tricky subject. Therefore it might be a good strategy to try to understand this first by looking at the simplest possible theory namely ϕ^4 theory. So consider again the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{1.784}$$

The whole issue of renormalisation concerns the following question:
What exactly do the parameters m and λ (and as we will see later also the field, here just ϕ) actually mean when the Lagrangian is used to make predictions for an experiment?

Naively we would just say that first we determine m and λ by performing two experiments which measure these parameters and then we can start predicting things by computing Feynman graphs and apply perturbation theory.

However, already here we are in trouble. Consider the exact vertex in ϕ^4:



To compare what we compute in perturbation theory (above) to the value of λ obtained in an experiment we see that the λ that appears in $\mathcal{L}$ is itself not the experimental value! This is clear since the tree-graph is given by the Feynman rule $-i\lambda$ but this is corrected by all the other terms in the above infinite series of terms. To solve this problem we will below introduce different λ parameters. But first we need to compute the first loop corrections drawn above to get a better feeling for the structure of the problem.

The series of terms above are given by the Feynman rules: $(k' = k + p_1 + p_2)$

$$i\mathcal{M}(p_1, p_2 \to p_3, p_4) = -i\lambda + (a) + (b) + (c) + \ldots \tag{1.785}$$

where

$$(a) = \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(k + p_1 + p_2)^2 - m^2 + i\epsilon}. \tag{1.786}$$

Here the factor $\frac{1}{2}$ comes from the symmetry factor $s = 2$ (two identical lines in the loop).

Define now $V(p^2)$ by

$$(a) := (-i\lambda)^2 iV(p^2)|_{p=p_1+p_2}. \tag{1.787}$$

This momentum is actually related to the Mandelstam variable $s = p^2 = (p_1 + p_2)^2$ and then we see that the expressions for the diagrams (b) and (c) are obtained by replacing s by the other Mandelstam variables $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$, respectively. The minus signs appear since p_3 and p_4 are out-going momenta.

Our next task is to analyse $V(p^2)$, that is, the four-dimensional momentum integral

$$V(p^2) := \frac{i}{2}\int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2+i\epsilon}\,\frac{1}{(k+p_1+p_2)^2-m^2+i\epsilon}. \qquad (1.788)$$

This analysis is hard to do directly in the above integral so a very useful trick is to Wick rotate! That is, we can let k^0 become complex and then turn the integral over the real part of k^0 (appearing in the integral above) into an integral along the imaginary k^0 axis by letting $k^0 \to ik^0$. This is called a **Wick rotation** and can be viewed as a rotation of the real axis into the imaginary one without passing any poles in the Feynman propagators.

The integral has now become Euclidean, with k^2 replaced by $-k_E^2$, and we can therefore introduce polar coordinates in this four-dimensional Euclidean momentum space by

$$\int_{-\infty}^{\infty}\frac{d^4k_E}{(2\pi)^4} = \int_0^{\infty} dk k^3 \int_{S^3} d\Omega_4, \qquad (1.789)$$

where k is now the radial coordinate in momentum space. The angular integral is over the unit radius 3-dimensional sphere S^3.

It is then a simple matter to check how $V(p^2)$ behaves for large momenta, i.e., in the UV limit. Introduce a large cut-off Λ in momentum space as follows

$$V(p^2) \propto \int^{\Lambda} dk\, k^3 \frac{1}{k^2}\frac{1}{k^2}, \qquad (1.790)$$

where we have taken Λ big enough so that masses and external momenta p_i can be neglected in the denominator of $V(p^2)$. Then

$$V(p^2) \propto \int^{\Lambda} dk\, k^3 \frac{1}{k^2}\frac{1}{k^2} \propto \int^{\Lambda}\frac{dk}{k} \propto \log\Lambda \to \infty \ \text{ as } \ \Lambda\to\infty, \qquad (1.791)$$

This is our first important result: the integral in $V(p^2)$ is divergent!

The above discussion and divergence analysis of $V(p^2)$ will force us to perform the following three steps:

1. Regularisation: This refers to the introduction of any kind of parameter (like Λ above) that can be used to define the way the integral approaches infinity when the parameter is taken to infinity (or to zero if that is how the divergence is emerging).

2. Renormalisation: This refers to the procedure required to relate the parameters in the Lagrangian to the measured values of these parameters. This step will also involve the fields themselves. The renormalisation needed here is multiplicative as we will see later.

3. Renormalisability: When the previous step is under control one can start checking if the theory is renormalisable. This is done by counting the divergent diagrams and comparing that number to the number of parameters in the Lagrangian (including the fields).

1. Regularisation: There are several ways to make a divergent integral convergent and physics must of course be independent of which one we use. This will be clear below. Here we will discuss three often used regularisations:
a) **Cut-off**: This is the one used above, that is after Wick rotation one introduces the cut-off parameter Λ by

$$\int_{-\infty}^{\infty} d^4k_E \to \int^{\Lambda} dk\, k^3 \int d\Omega, \tag{1.792}$$

which cuts of the integral at some large momentum Λ which is taken to infinity at the end. Note that this is an $SO(4)$ invariant procedure (corresponding to Lorentz invariance before Wick rotation) but it destroys gauge invariance in QED since in momentum space a gauge transformation reads $\delta A_\mu = ip_\mu\alpha$ and hence depends on the momentum. Thus it also affects unitarity.

b) **Pauli-Villars**: Here one introduces a heavy ghost particle of the same spin as the one in the divergent loop and then takes the mass M to infinity. Explicitly

$$D_F = \frac{i}{k^2 - m^2 + i\epsilon} \to \frac{i}{k^2 - m^2 + i\epsilon} - \frac{i}{k^2 - M^2 + i\epsilon}, \tag{1.793}$$

where the minus sign is the origin of the name *ghost* and heavy refers to the large value of M. This trick is Lorentz invariant and gauge invariant if applied to a photon propagator. It is not unitary until after the limit $M \to \infty$ is taken. The key point here is that for very large momenta where the masses can be neglected the two integrals cancel each other making the sum of the two terms UV finite. Of course, if $M \to \infty$ is taken first the second term is zero and we are back to the usual propagator.

c) **Dimensional regularisation**: Very nice to work with but physically a bit obscure perhaps. Here one generalises the momentum integrals to a general dimension d which does not even have to be integer:

$$d = 4 \to d = 4 - \epsilon \;\text{ where }\; \epsilon \to 0. \tag{1.794}$$

d) **Lattice regularisation**: Turning spacetime into a lattice makes it possible to compute certain quantities exactly. However, then the lattice must be removed by letting the lattice spacing go to zero which does not always work. Certain kind of chiral theories are also impossible to treat with this method. This will not be discussed any further in this course (see PS sect. 22.1).

The regularisation procedure raises a number of questions and it is therefore interesting to note that there are theories in four space-time dimensions which do not need this

step at all:
Field theory: Super-Yang-Mills with maximal number of supersymmetries known as $\mathcal{N} = 4$ SYM[41]. After this theory was discovered other less supersymmetric YM-theories have been found that are also finite.
String theory: Here there are no UV infinite diagrams at all at any loop order.

We now return to the integral $V(p^2)$ defined above which we found to be infinite. In order to compute it exactly (before Wick rotation) we need another trick due to Feynman: Introduce a **Feynman parameter** x by the following integral

$$\frac{1}{AB} = \int_0^1 dx \, \frac{1}{(xA + (1-x)B)^2}. \tag{1.795}$$

This is easily checked:

$$RHS = \left(-\frac{1}{xA + (1-x)B} \times \frac{1}{A-B}\right)_{x=0}^{x=1} = -\frac{1}{A-B}\left(\frac{1}{A} - \frac{1}{B}\right) = -\frac{1}{A-B} \times \frac{B-A}{AB} = \frac{1}{AB}. \tag{1.796}$$

Introducing this Feynman parameter into $V(p^2)$ while identifying A with $k^2 - m^2$ and B with $(k+p)^2 - m^2$ we get (the $+i\epsilon$ is not relevant here)

$$V(p^2) = \frac{i}{2}\int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{(x(k^2 - m^2) + (1-x)((k+p)^2 - m^2))^2}. \tag{1.797}$$

The expression in the denominator, often denoted D, can now be simplified somewhat (compare to PS p. 327):

$$D = x(k^2 - m^2) + (1-x)((k+p)^2 - m^2) = k^2 - m^2 + (1-x)(p^2 + 2k \cdot p). \tag{1.798}$$

Now we change integration variables from k to $l = k + (1-x)p$. This gives, still in Minkowski,

$$V(p^2) = \frac{i}{2}\int_0^1 dx \int \frac{d^4l}{(2\pi)^4} \frac{1}{(l^2 - \Delta)^2}, \text{ where } \Delta = m^2 - x(1-x)p^2. \tag{1.799}$$

This is a nice result since
1) V depends only p^2, not linearly on p^μ as before,
2) the l-integrand is independent of angles $\Rightarrow$ the Euclidean version of the integral is rather easy to compute exactly.

To compute $V(p^2)$ exactly we first Wick rotate: $l^0 := i\, l_E^0$ which implies

$$l^0 := i\, l_E^0 \Rightarrow l^2 = -l_E^2, \quad d^4l = i\, d^4l_E. \tag{1.800}$$

[41] This theory was proven finite to all loop orders using superspace Feynman diagrams in the following papers (in chronological order) L. Brink, O. Lindgren and B.E.W. Nilsson, Nucl. Phys B212 (1983), S. Mandelstam, Nucl. Phys B213 (1983) and L. Brink, O. Lindgren and B.E.W. Nilsson, Phys. Lett. B123 (1983).

Therefore the Euclidean version of $V(p^2)$ is

$$V(p^2)_E = -\frac{1}{2}\int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{1}{(l_E^2+\Delta)^2}, \text{ where } \Delta = m^2 - x(1-x)p^2. \tag{1.801}$$

Here we should note that p^μ is still in Minkowski space although we have Wick rotated in the integration variable l^0 to be able to perform the integral as swiftly as possible.

The next step is therefore to do the angular integrals. Here we will take another important step and get the result in any dimension d and then let d be any real positive number. How this is possible will become clear below.

First we split the whole Euclidean momentum integral into a radial part and an angular part by

$$\int d^d l_E = \int dl\, l^{d-1} \int_{S^{d-1}} d\Omega_d, \tag{1.802}$$

which is just a direct generalisation of the cases $d = 2$ and $d = 3$. The radial coordinate in Euclidean momentum space is denoted l (without any index E) and S^{d-1} is the $d-1$-dimensional unit sphere.

The angular part can be done as follows. Recall that $\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}$ and hence, for integer values of d,

$$\pi^{\frac{d}{2}} = (\sqrt{\pi})^d = \int d^d x\, e^{-x_1^2-x_2^2-\ldots.-x_d^2} = \int_0^\infty dr\, r^{d-1} e^{-r^2} \int d\Omega_d. \tag{1.803}$$

But here we integral over the radial coordinate r is rather easily done by setting $y = r^2$. Then

$$\int_0^\infty dr\, r^{d-1} e^{-r^2} = \frac{1}{2}\int_0^\infty dy\, y^{\frac{d}{2}-1} e^{-y}. \tag{1.804}$$

This integral is quite remarkable since for $d = 2$ it becomes

$$\int_0^\infty dy\, e^{-y} = [-e^{-y}]_0^\infty = 1. \tag{1.805}$$

Denoting the integral $\Gamma(d/2)$ for now we have $\Gamma(1) = 1$. By partial integrations one can prove that

$$\Gamma(n+1) = n\Gamma(n). \tag{1.806}$$

This is recursive and can be expressed as

$$\Gamma(n) = (n-1)!, \tag{1.807}$$

so this function is the standard *Gamma* function, here represented by the integral above.

However, the integral representation is not restricted to integer values of the argument n so we can define the angular integral above for any dimension d even when d is not an integer. Thus, for any real d, we have

$$\int d\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}. \tag{1.808}$$

To check this result, recall that $\Gamma(1/2) = \sqrt{\pi}$ and thus $\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$:

$$S^1 : \int d\Omega_2 = \frac{2\pi^{\frac{2}{2}}}{\Gamma(\frac{2}{2})} = 2\pi, \ \ S^2 : \int d\Omega_3 = \frac{2\pi^{\frac{3}{2}}}{\Gamma(\frac{3}{2})} = \frac{2\pi^{\frac{3}{2}}}{\frac{1}{2}\pi^{\frac{1}{2}}} = 4\pi. \tag{1.809}$$

For the case we are interested in here, that is $d = 4$, we get

$$S^3 : \int d\Omega_4 = \frac{2\pi^{\frac{4}{2}}}{\Gamma(\frac{4}{2})} = \frac{2\pi^2}{\Gamma(2)} = 2\pi^2. \tag{1.810}$$

Finally, we can summarise these results in the formula

$$\int d^d l_E = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \int_0^\infty dl\, l^{d-1}. \tag{1.811}$$

1. Regularisation: Now we can compute the integrals that appear in these loop corrections after Wick rotation:

$$I_l = \int_0^\infty dl\, l^{d-1} \frac{1}{(l^2+\Delta)^2} = (\text{set}\, y = l^2) = \frac{1}{2}\int_0^\infty dy\, y^{\frac{d}{2}-1} \frac{1}{(y+\Delta)^2}. \tag{1.812}$$

Set now

$$x = \frac{\Delta}{y+\Delta} \Rightarrow dx = -\frac{\Delta\, dy}{(y+\Delta)^2} \Rightarrow \frac{dy}{(y+\Delta)^2} = -\frac{dx}{\Delta}, \tag{1.813}$$

and solving for y we get

$$x = \frac{\Delta}{y+\Delta} \Rightarrow y = \frac{\Delta}{x} - \Delta = \Delta(\frac{1}{x} - 1) = \Delta\frac{1-x}{x}. \tag{1.814}$$

Using these relations to turn the integral into an x-integral we get

$$I_l = \frac{1}{2}\int_0^1 \frac{dx}{\Delta}\, \Delta^{\frac{d}{2}-1} x^{1-\frac{d}{2}} (1-x)^{\frac{d}{2}-1} = \frac{1}{2}\Delta^{\frac{d}{2}-2} \int_0^1 dx\, x^{1-\frac{d}{2}}(1-x)^{\frac{d}{2}-1}. \tag{1.815}$$

In the spirit of the integral representation of the Γ function above one can now also express this integral in terms of Γ functions. The relation is provided by the following definition of the *Beta* function $B(\alpha, \beta)$:

$$B(\alpha,\beta) := \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{1.816}$$

Thus we have

$$I_l = \frac{1}{2}\Delta^{\frac{d}{2}-2} B(2-\tfrac{d}{2}, \tfrac{d}{2}) = \frac{1}{2}\Delta^{\frac{d}{2}-2} \frac{\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(2)}. \tag{1.817}$$

This is a very interesting result: it is divergent for $d = 4$ due to $\Gamma(2 - \frac{d}{2}) = \Gamma(0) = \infty$. In fact, by analytic continuation one can show that $\Gamma(x)$ is finite for all real values of x except at non-positive integer values. To define this divergence in the physics problem we are looking at here, we let the dimension d become slightly less than 4, i.e. instead of using $d=4$ we insert $d=4-\epsilon$. This gives

$$\Gamma(2-\tfrac{d}{2}) = \Gamma(\frac{\epsilon}{2}) \approx \frac{2}{\epsilon} + \gamma + \mathcal{O}(\epsilon), \tag{1.818}$$

where we in the last step used a well-known expansion of the Γ function for small arguments. The constant γ is the Euler-Mascheroni constant $\gamma \approx 0.5772...$

At this point we should return to the question how physics can be extracted from these formulas. To do this we go back to the momentum integral in $V(p^2)$ which now is in d dimensions, and in the limit $\epsilon \to 0$ becomes

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l^2+\Delta)^2}|_{d=4-\epsilon} = \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \log\Delta - \gamma + \mathcal{O}(\epsilon)\right). \tag{1.819}$$

There are a couple of steps before one finds this result. The derivation goes as follows:

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l^2+\Delta)^2} = \int \frac{d\Omega_d}{(2\pi)^d} \int_0^\infty dl\, l^{d-1} \frac{1}{(l^2+\Delta)^2} = \frac{1}{(2\pi)^d} \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \frac{1}{2} \Delta^{\frac{d}{2}-2} \frac{\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(2)}. \tag{1.820}$$

This can be simplified a bit to, using also $\Gamma(2)=1$,

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l^2+\Delta)^2} = \frac{\pi^{\frac{d}{2}}}{(2\pi)^d} \Delta^{\frac{d}{2}-2} \Gamma(2-\tfrac{d}{2}). \tag{1.821}$$

To get the result quoted above we need not only the expansion of the last Γ factor given above, but also to expand $\Delta^{\frac{d}{2}-2}$ for small ϵ: with $d=4-\epsilon$ we get

$$\Delta^{\frac{d}{2}-2} = \Delta^{-\frac{\epsilon}{2}} = e^{-\frac{\epsilon}{2}\log\Delta} = 1 - \tfrac{\epsilon}{2}\log\Delta + \mathcal{O}(\epsilon^2). \tag{1.822}$$

Multiplying these two expansions together and collecting the $1/\epsilon$ and the ϵ independent terms gives the result above. Note that also the 2π factors could have been expanded like this and then contributed to the finite constant (momentum independent) terms. As will be clear later such terms contain no physics information which is why we skipped those terms here.

In fact, this is the point where we should stop and clarify where the physics information come from in the above expression

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l^2+\Delta)^2}|_{d=4-\epsilon} = \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \log\Delta - \gamma + \mathcal{O}(\epsilon)\right). \tag{1.823}$$

There are three kinds of terms here that survive the limit $\epsilon \to 0$: The first divergent one, the second finite but p^2 dependent one, and the third one which is just a finite constant. The renormalisation procedure to be discussed later when we fully understand the regularisation step discussed here will show that all information about the physics is contained in the $\log\Delta(p^2)$ term basically because of its p^2 dependence.

Having stated this fact we can continue to check if the other regularisation procedures generate the same physics. So let us complete the calculation with the cut-off parameter Λ. Then we need to compute, for $d = 4$,

$$I_l(\Lambda) = \int_0^{\Lambda} dl\, l^{d-1} \frac{1}{(l^2+\Delta)^2} = (y = l^2) = \frac{1}{2}\int_0^{\Lambda^2} dy\, y^{\frac{d}{2}-1} \frac{1}{(y+\Delta)^2}$$

$$= (d=4) = \frac{1}{2}\int_0^{\Lambda^2} \frac{dy\, y}{(y+\Delta)^2} = \frac{1}{2}\int_0^{\Lambda^2} dy\, y \partial_y(-\frac{1}{y+\Delta}) = \frac{1}{2}\int_0^{\Lambda^2} dy \frac{1}{y+\Delta} + \frac{1}{2}\left(-\frac{y}{y+\Delta}\right)|_0^{\Lambda^2}$$

$$= \frac{1}{2}\log\frac{\Lambda^2+\Delta}{\Delta} - \frac{1}{2}\frac{\Lambda^2}{\Lambda^2+\Delta}. \tag{1.824}$$

In the limit $\Lambda \to \infty$ this reduces to

$$I_l(\Lambda \to \infty) = \log\Lambda - \frac{1}{2}\log\Delta - \frac{1}{2}, \tag{1.825}$$

and hence

$$\int \frac{d^d l_E}{(2\pi)^d}\frac{1}{(l^2+\Delta)^2}|_{\Lambda\to\infty} = \frac{1}{(4\pi)^2}(-\log\Delta + 2\log\Lambda - 1). \tag{1.826}$$

Recalling the rule stated above about which term contains the physics information, namely the one depending on p^2, i.e. $\log\Delta(p^2)$, we find that the physics is the same as for dimensional regularisation.

As a last case we also do the computation with Pauli-Villars regularisation. Thus, with the mass dependence in $\Delta(m) = m^2 - x(1-x)p^2$, we have

$$I_l(M) = \int_0^{\infty} dl\, l^3 \left(\frac{1}{(l^2+\Delta(m))^2} - \frac{1}{(l^2+\Delta(M))^2}\right) \tag{1.827}$$

which by setting $y = l^2$ becomes

$$= \frac{1}{2}\int_0^{\infty} dy\, y \left(\frac{1}{(y+\Delta(m))^2} - \frac{1}{(y+\Delta(M))^2}\right)$$

$$= \frac{1}{2}\int_0^{\infty} dy \left(\frac{1}{y+\Delta(m)} - \frac{1}{y+\Delta(M)}\right) - \frac{1}{2}\left(\frac{y}{y+\Delta(m)} - \frac{y}{y-\Delta(M)}\right)|_0^{\infty}$$

$$= -\frac{1}{2}\log\frac{\Delta(m)}{\Delta(M)} = -\frac{1}{2}\log\Delta(m) + \frac{1}{2}\log\Delta(M). \tag{1.828}$$

Thus

$$\int \frac{d^d l_E}{(2\pi)^d}\frac{1}{(l^2+\Delta)^2}|_{M\to\infty} = \frac{1}{(4\pi)^2}(-\log\Delta(m) + \Delta(M)), \tag{1.829}$$

which once again provides the same physics information stored in the term $-\frac{1}{(4\pi)^2}\log\Delta(m)$ while the divergence is captured by the other term $\frac{1}{(4\pi)^2}\log\Delta(M)$.

We now turn to the issue of renormalisation which will explain the above statement about where to find the physics information after regularisation.

2. Renormalisation:

1.1.2 Renormalisation in ϕ^4 theory

Consider again the Lagrangian for ϕ^4 theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{1.47}$$

As already mentioned the whole issue of renormalisation concerns the following question:

Question: What exactly do the parameters m and λ actually mean when the Lagrangian is used to make predictions for an experiment? As we will see later this question also involves the field, here just ϕ.

Idea: Choose the parameters in $\mathcal{L}$ so that the observable quantities take their physical (finite) values.

How is this done? Considered again the scattering process discussed above

$$i\mathcal{M}(12 \to 34) = -i\lambda + (-i\lambda)^2(iV(s) + iV(t) + iV(u)) + ... \tag{1.48}$$

where, with $\Delta = m^2 - x(1-x)p^2$,

$$V(p^2) = -\frac{1}{32\pi^2}\int_0^1 dx \left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log(m^2 - x(1-x)p^2)\right) \tag{1.49}$$

and the dots indicate an infinite series of higher loop terms (at higher and higher order in the coupling constant λ).

Note now the following facts:

1) λ is not the physical coupling constant λ_{phys} measured in experiments since that value is determined by the whole perturbation series, i.e.,

$$i\mathcal{M} = -i\lambda_{phys}(p^2), \tag{1.50}$$

which does depend on p^2, the momentum at which the scattering experiment is performed. This p^2 dependence is seen in experiments so it is not surprising that also the theory, via $V(p^2)$, indicates that p^2 plays a role here. We will state this fact as

$$\lambda_{measured} = \lambda_{phys}(p^2). \tag{1.51}$$

We emphasise here that the Lagrangian $\mathcal{L}(x)$ can NOT contain parameters that depend on momenta since that would make it non-local or worse.

2) Our intuition that coupling constants are constant come from experiments at very low energies and therefore it is natural to define "the coupling constant" λ at zero 3-momentum $\mathbf{p} = 0$, called the **subtraction point**, by

$$\lambda := \lambda_{phys}(p^\mu = (m, 0, 0, 0)). \tag{1.52}$$

This fixed number λ determined by experiment is the value we will use at the end in the so called **renormalised Lagrangian**.

The standard notation used in the QFT literature is to denote quantities like the mass and the coupling constant appearing in the Lagrangian as λ_0 and m_0, called as the **bare** quantities since λ and m will, as just noted above, refer to the finite physical (i.e., measured) values at the subtraction point. This is just a renaming of the parameters in $\mathcal{L}$!

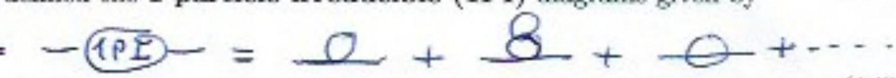
So let us restart the discussion this time from the Lagrangian with the bare parameters

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4. \tag{1.53}$$

Now we need to discuss also the role of the field ϕ and at the same time the mass m_0. To do this we consider the exact two-point function, i.e., the propagator, in perturbation theory:

$$\begin{aligned} &= \frac{i}{p^2 - m_0^2 + i\epsilon} + \frac{i}{p^2 - m_0^2 + i\epsilon}(-iM(p^2))\frac{i}{p^2 - m_0^2 + i\epsilon} \\ &+ \frac{i}{p^2 - m_0^2 + i\epsilon}(-iM(p^2))\frac{i}{p^2 - m_0^2 + i\epsilon}(-iM(p^2))\frac{i}{p^2 - m_0^2 + i\epsilon} + \ldots \end{aligned} \tag{1.54}$$

where we have defined the **1-particle irreducible (1PI)** diagrams given by

$-iM(p^2) =$ 

(1.55)

Then the exact propagator is a geometric series that can be summed up, provided the $M(p^2)$ is small enough, to give

$$\frac{i}{p^2 - m_0^2}\left(\frac{1}{1 - \frac{M(p^2)}{p^2 - m_0^2}}\right) = \frac{i}{p^2 - m_0^2 - M(p^2) + i\epsilon}. \tag{1.56}$$

As we have seen already the quantity $M(p^2)$ will be infinite if not regularised (but always close to infinite), and so will m_0 but in a way that cancels in the combination in the propagator $m_0^2 - M$. This is not an acceptable situation (e.g., in summing up the geometric series). This problem is eliminated in the renormalised Lagrangian that will be defined below!

This form of the exact propagator has two important implications:

1) The physical mass of the particle is determined by the value of p^2 where the propagator diverges, that is for the p^2 that solves the equation

$$p^2 - m_0^2 - M(p^2) = 0. \tag{1.57}$$

Call this value m^2, that is by definition $p^2 = m^2$. This is the mass we obtain from an experiment and this value of p^2 plays the role of subtraction point in this case.

2) Let us now expand $M(p^2)$ around this physical value m^2:

$$M(p^2) = M(m^2) + (p^2 - m^2)\left(\frac{d}{dp^2}M(p^2)\right)|_{m^2} + ... \tag{1.58}$$

Inserting this expansion into the propagator above gives

$$\frac{i}{p^2 - m_0^2 - M(p^2) + i\epsilon} = \frac{i}{p^2 - m_0^2 - (M(m^2) + (p^2 - m^2)(\frac{d}{dp^2}M(p^2))|_{m^2} + ...) + i\epsilon}. \tag{1.59}$$

This means that for p^2 close to the pole $p^2 - m_0^2 - M(m^2) = p^2 - m^2$ and the propagator reads

$$\frac{i}{(p^2 - m^2)(1 - \frac{d}{dp^2}M(p^2))|_{m^2}} := \frac{iZ}{p^2 - m^2}, \tag{1.60}$$

where we can identify the **field renormalisation constant** Z as

$$Z := \left(1 - \frac{d}{dp^2}M(p^2)|_{m^2}\right)^{-1}. \tag{1.61}$$

Since the exact propagator is really the two-point function $\langle\Omega|T\phi^{exact}(x)\phi^{exact}(y)|\Omega\rangle$ we want it to behave as $\frac{i}{p^2-m^2+i\epsilon}$ close the pole where m is the physical mass. It is therefore convenient to define a new field ϕ_r, the **renormalised** field, by rescaling the field in the Lagrangian ϕ as follows

$$\phi = \sqrt{Z}\phi_r. \tag{1.62}$$

The Lagrangian is then written as

$$\mathcal{L} = \frac{1}{2}Z\partial_\mu\phi_r\partial^\mu\phi_r - \frac{1}{2}Z\,m_0^2\phi_r^2 - \frac{\lambda_0}{4!}\,Z^2\,\phi_r^4, \tag{1.63}$$

and now the key point about this version of the Lagrangian is that the exact propagator close to the pole is precisely

$$\langle\Omega|T\phi_r(x)\phi_r(y)|\Omega\rangle = \frac{i}{p^2 - m^2 + i\epsilon} + ... \tag{1.64}$$

Now one could start doing perturbation theory using the standard Feynman rules but expressed in terms of the new renormalised field ϕ_r and the bare constants m_0 and λ_0. However, there is a much more convenient way to view this Lagrangian which emerges if one defines the following δ-parameters:

$$\delta_Z := Z - 1,\;\; \delta_m := m_0^2 Z - m^2,\;\; \delta_\lambda := \lambda_0 Z^2 - \lambda. \tag{1.65}$$

Expressing the Lagrangian in terms of these δ parameters and the physical quantities m, λ and the field ϕ_r, instead of the bare parameters m_0 and λ_0 together with ϕ has no physical effect but gives the renormalisation procedure a very appealing structure as we will see. Remember that λ and m are just fixed numbers so these equations are just replacing Z, λ_0, m_0 by their respective δ parameters $\delta_Z, \delta_\lambda, \delta_m$.

This change of parameters gives the Lagrangian the following form:

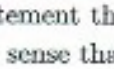
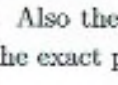
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_r\partial^\mu\phi_r - \frac{1}{2}m^2\phi_r^2 - \frac{\lambda}{4!}\phi_r^4 \tag{1.66}$$

$$+\frac{1}{2}\delta_Z\partial_\mu\phi_r\partial^\mu\phi_r - \frac{1}{2}\delta_m m^2\phi_r^2 - \frac{1}{4!}\delta_\lambda\phi_r^4. \tag{1.67}$$

The fundamental interpretation of this **renormalised** Lagrangian is the following:

1) The first line of terms generate the standard Feynman rules and look exactly the same as they did in the beginning but the parameters are now the physical measured ones, a statement that will be made more precise below. Also the field ϕ^r is the physical one in the sense that it gives the physical behaviour of the exact propagator close to the pole.

2) The second line contains the so called **counter terms** and they give rise to a new set of Feynman rules:

 $= i(\delta_Z\, p^2 - \delta_m),$  $= -i\delta_\lambda.$ (1.68)

Note that the first term on the second line in $\mathcal{L}$ looks like a kinetic term but that we are now interpreting it as an interaction term!

The huge advantage of this renormalised Lagrangian over any other equivalent form will be obvious when we start using it. However, we still need to be precise about what the physical parameters m and λ in $\mathcal{L}$ are. The discussion in the beginning about the physical coupling constant suggest what to do. We should, quite arbitrarily, pick a momentum, called the **subtraction point(s)**, at which we decide the measured value to be the one that is given to the m and λ appearing in $\mathcal{L}$. Such a choice is necessary for at least two reasons:
1) The measured values of these parameters depend on p^2,
2) The parameters in the Lagrangian must be independent of p^2.

Let us now apply these ideas in a couple of explicit cases to see how it works. First we return to the scattering $12 \to 34$ in ϕ^4 theory. We saw above that the tree diagram is corrected at one-loop level by three diagrams represented by a function $V(p^2)$ where $p^2 = s, t, u$, respectively, for the three different channels. In renormalised ϕ^4 theory there is one more diagram coming from the new interaction vertex $-i\delta_\lambda$. Thus we have now the diagram expansion

which gives the scattering amplitude

$$i\mathcal{M}(12 \to 34) = -i\lambda + (-i\lambda)^2(iV(s) + iV(t) + iV(u)) - i\delta_\lambda. \tag{1.69}$$

The first thing to do is to determine the constant (i.e., p^2 independent) value of δ_λ. This is done by demanding that at the subtraction point, which in case we choose to be $\mathbf{p_1} = \mathbf{p_2} = 0$, the counter term cancels the one-loop terms so that

$$\mathcal{M}(12 \to 34)|_{subt.point} = -i\lambda. \tag{1.70}$$

In other words: at the subtraction point the physically measured value of the coupling constant, i.e. $\mathcal{M}(12 \to 34)|_{subt.point}$, is the value we give the coupling constant, λ, in the Lagrangian.

With this choice of subtraction point we have $s = (p_1 + p_2)^2 = 4m^2$. Then since $s + t + u = 4m^2$ we also have that $t + u = 0$ which implies $E_3 + E_4 = 2m$, and thus also that $\mathbf{p_3} = \mathbf{p_4} = 0$. The subtraction point is therefore given by $s = 4m^2$ and $t = u = 0$. Thus

$$\delta_\lambda = -\lambda^2(V(4m^2) + 2V(0)). \tag{1.71}$$

Using this result we can finally write down the scattering amplitude for any momenta p_1 and p_2:

$$i\mathcal{M} = -i\lambda - \frac{i\lambda^2}{32\pi^2}\int_0^1 dx \left(\log\frac{m^2 - x(1-x)s}{m^2 - x(1-x)4m^2} + \log\frac{m^2 - x(1-x)t}{m^2} + \log\frac{m^2 - x(1-x)u}{m^2}\right). \tag{1.72}$$

This formula explains all the subtle features of renormalised perturbation theory:
1) The Lagrangian is well-defined since the coupling constant λ appearing in it is a constant, whose value is exactly the one measured at the subtraction point: at the subtraction point the above equation becomes $i\mathcal{M} = -i\lambda$.
2) The measured value of the coupling constant at general momenta, λ_{phys}, is the value of $i\mathcal{M}(p^2)$ which can be computed in perturbation theory as done here to first loop-order.
3) The counter term is a sum of constant pieces, finite or infinite, at each power in λ such that they exactly cancel the corresponding terms that arise in the loop calculations. All kinds of regularisation parameters can then be eliminated (i.e., taken to infinity or zero) leaving only finite results.

Comment: In QED the analogues calculation can be done and compared to experiment and the renormalised theory is found to work extremely well. In other words, as suggested by renormalised QED the electric charge e is not a constant but depends on the energy scale at which the experiment is performed. This is also exactly what is seen in experiments: in terms of the fine structure constant which takes it usual value 1/137 at low energy (or large scale) there is a 5 per cent increase in its value going from the subtraction point at low energy to 30 GeV. This fact will be given a quite intuitive explanation in the very last lecture.

Having understood how renormalised perturbation theory works for the coupling constant let us turn to the slightly more complicated case of the renormalisation of the mass m and the field ϕ. The theory now gives the $1PI$ blob as

which gives schematically without precise symmetry factors etc (this is M not $\mathcal{M}$)

$$-iM(p^2) = -i\lambda\frac{1}{2}\int\frac{d^4k}{k^2} + (-i\lambda)^2\int\frac{d^2k_1 d^4k_2}{k_1^2(k_2^2)^2} + (-i\lambda)^2\int\frac{d^2k_1 d^4k_2}{k_1^2k_2^2(p-k_1-k_2)^2} + ... \quad (1.73)$$

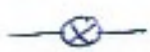
Let us first compute the snail-diagram at order λ using the standard methods. Wick rotation $k^0 \to ik^0$, $d^dk \to id^dk$ and $k^2 - m^2 \to -(k_E^2 + m^2)$, gives

$$-\frac{i\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2} \to -\frac{i\lambda}{2}\int\frac{d^dk_E}{(2\pi)^d}\frac{1}{k_E^2+m^2} = -\frac{i\lambda}{2(2\pi)^d}\int dk\, k^{d-1}\frac{1}{k^2+m^2}\int d\Omega_d \quad (1.74)$$

$$= -\frac{i\lambda}{2}\frac{(m^2)^{\frac{d}{2}-1}}{(4\pi)^{\frac{d}{2}}}\Gamma(1-\tfrac{d}{2}). \quad (1.75)$$

To this result for the snail graph we must now add the counter term

$$\text{—}\!\otimes\!\text{—} \quad = i(p^2\delta_Z - \delta_m), \quad (1.76)$$



and demand that at the subtraction point, being $p^2 = m^2$ in this case (i.e., not the same as for the vertex),

$$M(p^2)|_{p^2=m^2} = 0, \quad \frac{d}{dp^2}M(p^2)|_{p^2=m^2} = 0, \quad (1.77)$$

which implies

$$\delta_Z = 0, \quad \delta_m = -\frac{\lambda}{2}\frac{(m^2)^{\frac{d}{2}-1}}{(4\pi)^{\frac{d}{2}}}\Gamma(1-\tfrac{d}{2}). \quad (1.78)$$

Since the snail graph is p^2 independent the end result is that the counter term cancels out the snail graph completely. Note that this renormalisation also involves the cancellation of an infinite piece coming from $\Gamma(1-\frac{d}{2})$. In fact, the snail graph is Λ^2 divergent which in dimensional regularisation translates to

$$\Gamma(1-\tfrac{d}{2})|_{d=4-\epsilon} = \Gamma(-1+\tfrac{\epsilon}{2}) = \frac{\Gamma(\frac{\epsilon}{2})}{-1+\frac{\epsilon}{2}}. \quad (1.79)$$

Expanding these two factors for small ϵ we get

$$\Gamma(1-\tfrac{d}{2})|_{d=4-\epsilon} = -\left(\tfrac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)\right)\left(1 + \tfrac{\epsilon}{2} + \mathcal{O}(\epsilon^2)\right) = -\left(\tfrac{2}{\epsilon} - \gamma + 1 + \mathcal{O}(\epsilon)\right), \quad (1.80)$$

which is infinite but has no term relevant for physics and hence it gets completely cancelled by the counter term.

which is infinite but has no term relevant for physics and hence it gets completely cancelled by the counter term.

Thus the interesting points here are

1. The first (the snail) is infinite (goes as Λ^2) but independent of p^2.
2. The same is true for the second term (the double-snail).
3. The third term (the sunset) is also infinite (goes also as Λ^2) but does depend on p^2.

Let us investigate the third term a bit more. Let us call the function this diagram generates $f(p^2)$. The following argument is, in fact, applicable even if this function represents the entire series of perturbation terms. The integral in the third term is as we saw above

$$f(p^2) := (-i\lambda)^2 \int \frac{d^4k_1 d^4k_2}{k_1^2 k_2^2 (p-k_1-k_2)^2}. \tag{1.81}$$

We know from the Feynman parameter trick (together with a shift in the integration variables) that this integral can be made to depend only on p^2 and not linearly onp^μ. Let us then Taylor expand this function around the subtraction point $p^2 = m^2$:

$$f(p^2) = f(m^2) + p^2 \left(\frac{d}{dp^2} f(p^2) \right)|_{p^2=m^2} + \frac{1}{2}(p^2)^2 \left((\frac{d}{dp^2})^2 f(p^2) \right)|_{p^2=m^2} + ... \tag{1.82}$$

The interesting thing that happens here is that the degree of divergence of the integral decreases with each extra p^2 derivative:

$$\frac{d}{dp^2} \int \frac{d^4k_1 d^4k_2}{k_1^2 k_2^2 (p-k_1-k_2)^2} \propto \int \frac{d^4k_1 d^4k_2}{k_1^2 k_2^2 (p-k_1-k_2)^4}. \tag{1.83}$$

Thus the derivative turns the Λ^2 divergent integral on the LHS into the $\log \Lambda$ divergent integral on the RHS. Doing another derivative will therefore produce a convergent integral that goes as Λ^{-2} as $\Lambda \to \infty$. Thus the sunset graph gives rise to (as would also the entire series of terms) two infinite constants at order λ^2 that must be cancelled in the subtraction procedure. We express this as follows

$$f(p^2)_{snail}|_{div} \propto \Lambda^2 + p^2 \log \Lambda + finite. \tag{1.84}$$

Both divergent term will show up in dimensional regularisation as simple poles in ϵ and can thus be cancelled by adding new infinite terms at order λ^2 to the counter terms δ_m and δ_Z where the latter one does indeed multiply p^2 in the counterterm. This cancellation procedure can in principle be carried out to arbitrary order in perturbation theory and the δ-parameters will therefore be infinite power series in the coupling constant λ.

The fundamental question that must now be asked is: What happens if there are infinite Feynman graphs appearing in the scattering process $2 \to 4$ which does not correspond to a term in the Lagrangian and hence cannot be cancelled against a counter term? The only way out of this dilemma is to add the corresponding interaction term to the case: once one adds one single non-renormalisable term an infinite set of higher interaction terms must be added and the whole theory becomes non-renormalisable in the sense that an infinite number of experiments must be performed before a prediction can be made. Our next task is therefore to find a way to identify these dangerous terms that will render a theory useless in this sense.

Comment: This situation should be compared to what happens in gravity which is non-renormalisable but still a very useful theory! Recall that the Einstein-Hilbert Lagrangian $\mathcal{L}_{EH} = \sqrt{-\det g}R$ contains both the metric $g_{\mu\nu}$ and its inverse so if one expands it in terms of $h_{\mu\nu}$ defined by $g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{8\pi G}h_{\mu\nu}$ (using standard GR conventions with $g_{\mu\nu}$ the curved metric and $\eta_{\mu\nu}$ the flat Minkowski one) the Lagrangian becomes an infinite series of terms in powers of $h_{\mu\nu}$ all with two derivatives and the indices contracted in more and more complicated ways. However complicated this $\mathcal{L}_{EH}$ is in this expansion around Minkowski space its first term is just a conventionally normalised kinetic one $\frac{1}{2}(\partial_\mu h_{\nu\rho})^2$ while the next one is schematically $\sqrt{8\pi G}h\partial h\partial h$ etc for the following higher order terms. Thus the coupling constant in Einstein's general relativity is $\sqrt{8\pi G}$ which has dimension L^1 making the theory non-renormalisable in the sense defined above.

1.1.3 Renormalisability of ϕ^4 theory

In order to discuss the issue of whether a theory is renormalisable or not we must first investigate where, that is in which Feynman diagrams, divergencies occur. Then the question is whether all divergencies can be cancelled by good counter terms, that is counter terms that do not lead to even more divergencies. We will do this in two steps:

1. Count divergencies.
2. Check renormalisability.

In recent years there has been a development towards a slightly different attitude to renormalisibility and its role in defining consistent quantum field theories. This involves so called *effective field theories* and will be discussed a bit at the end of the course if time permits.

Counting divergencies in d-dimensional ϕ^n: Consider again the Lagrangian for the ϕ^4 but now generalised to any dimension d, i.e., d can be 2,3,4,5 or any other integer number, and with an order n interaction term ϕ^n (for some positive integer n):

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{n!}\phi^n. \tag{1.85}$$

For a general Feynman diagram we denote the number of the different parts as follows:

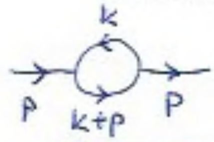
$$\begin{aligned} &N \text{ number of external legs,} \\ &V \text{ number of verticies,} \\ &P \text{ number of propagators} \\ &L \text{ number of loops.} \end{aligned} \tag{1.86}$$

Then we define the **superficial degree of divergence**, denoted D, by

$$D := dL - 2P, \tag{1.87}$$

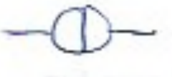
since each loop contributes the dimension of d^dk (i.e., d) and each propagator the dimension of $1/k^2$ (i.e., -2) to the total (mass=momentum=$1/L$) dimensionality of the integral. This definition is exactly what we used in the beginning when discussing the dependence of loop integrals in ϕ^4 theory on the cut-off parameter Λ. If D turns out to be non-negative the integral diverges as Λ^D for positive D, or $\log\Lambda$ if $D=0$. The name "superficial" is used because there are complicated Feynman diagrams where D does not give the whole story. We will only discuss one such case, namely gauge theory in the form of QED[2].

First some examples from ϕ^3 theory in six dimensions:

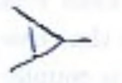


$$= \int^\Lambda d^6k\,\frac{1}{k^2}\,\frac{1}{(k+p)^2} \propto \Lambda^2, \text{ OK since } D = 6\cdot 1 - 2\cdot 2 = 2. \tag{1.88}$$

[2]Other situations where this must be analysed with some care arise at higher loop orders. Examples are discussed in PS sections 10.4 and 10.5. These two sections are not part of this course.



$$= \int^\Lambda d^6k_1 \int^\Lambda d^6k_2 \, (\frac{1}{k^2})^5 \propto \Lambda^2, \;\; \text{OK since } D = 6\cdot 2 - 2\cdot 5 = 2. \tag{1.89}$$



$$= \int^\Lambda d^6k \, (\frac{1}{k^2})^3 \propto \log\Lambda, \;\; \text{OK since } D = 6\cdot 1 - 2\cdot 3 = 0. \tag{1.90}$$

$$= \int^\Lambda d^6k \, (\frac{1}{k^2})^4 \propto \Lambda^2, \;\; \text{finite, OK since } D = 6\cdot 1 - 2\cdot 4 = -2. \tag{1.91}$$

These examples are quite trivial but we can learn a few things lot from them:

–$D = 2$ seems to be the maximal value.
–D decreases with increased number of external legs.
–Adding a new internal line (propagator) adds $6 - 2\cdot 3 = 0$ to D, i.e., the number of vertices does not matter so D is the same to all orders in perturbation theory (i.e., in powers of λ): D only depends on the number N of external legs of the diagram.

All these conclusions are also true for ϕ^4 theory in four dimensions but not if either d or n is changed in these cases (i.e., $d = 6$ is tied to ϕ^3 and $d = 4$ is tied to ϕ^4 for these conclusions to be true). It is rather easy to derive a general formula for D which contains these facts as special cases.

To do this we make use of two relations between N, V, P and L defined above. First:

$$nV = N + 2P. \tag{1.92}$$

This is a direct consequence of the fact that each external leg connects one end and each internal line connects two ends to the available vertices in the Feynman diagram. Second:

$$L = P - V + 1. \tag{1.93}$$

This relation follows by counting the total number momentum integrals and momentum delta-functions $\delta^d(\Sigma p)$ using x-space Feynman rules: The momentum integrals all come from the propagators $D_F(x-y) = \int d^dp \, e^{ip\cdot(x-y)} \frac{i}{p^2-m^2}$ but some of them can be done using the delta functions coming from the vertex integrals $\int d^dz_i \Rightarrow \delta^d(\Sigma p)$. The ones left at the end are the loop momenta. Thus we have $L = P - (V-1)$ where the 1 refers to the $\delta^d(\Sigma p)$ implementing overall momentum conservation that is not used and hence remains in the final expression.

Using these two relations to eliminate first L and then P from $D = dL - 2P$ we find:

$$(\phi^n)|_d: \quad D = d + \left(n\frac{d-2}{2} - d\right)V - \frac{d-2}{2}N. \tag{1.94}$$

This is an very nice formula containing a lot of information:
1. Increasing $N \Rightarrow$ decreasing D for $d > 2$ (as we saw above).
2. $d = 2$ is very special: $D = 2 - 2V$ independent of N.
3. In d dimensions a scalar field has length dimension $[\phi] = L^{-\frac{d-2}{2}}$. Thus the coupling constant λ_n from the interaction term $\frac{\lambda_n}{n!}\phi^n$ has dimension $[\lambda_n] = L^{n\frac{d-2}{2}-d}$. Thus we see

that the two special cases analysed above $(d,n) = (6,3)$ and $(d,n) = (4,4)$, both have dimensionless coupling constants since $d = n\frac{d-2}{2}$ in both cases. Therefore we have that

$$n = \frac{2d}{d-2} \Rightarrow D = d - \frac{d-2}{2}N, \tag{1.95}$$

and the conclusions we found above by looking at these special cases follow directly. There is a third quite interesting case of this kind, namely $(d,n) = (3,6)$[3].

Comment: In all these cases with a dimensionless coupling constant the massless theory has more space-time symmetry than Poincaré: They are invariant under the conformal group. The conformal group is the Poincaré plus scale transformations $x^\mu \to \Omega x^\mu$, which is a symmetry of the light-cone $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = 0$. Thus Maxwell's theory is conformal as well as QED with massless fermions (since e is dimensionless), and, in fact, the whole of the standard model is conformal before the Higgs effect if we drop the mass term for the Higgs field. Gravity is not conformal.

Consider now a scalar theory in d dimensions with a ϕ^p interaction term where $p < n$. Then

$$D = d + \left(p\frac{d-2}{2} - d\right)V - \frac{d-2}{2}N, \text{ where } p < n = \frac{2d}{d-2}, \tag{1.96}$$

which implies that: The bracket is *negative* and hence D decreases with increasing number of vertices V (for $d > 2$).
Note: If instead $p > n$ then the bracket is positive and D will become positive (with new infinite diagrams appearing) for large enough V for any N.

The condition $n = \frac{2d}{d-2}$ on the power of the interaction (giving a dimensionless coupling constant) is thus the boarder line case between ϕ^p with $p < n$ and $p > n$. It gives rise to the following classification of scalar field interactions and, in fact, theories in general:

Finite: Has no infinite Feynman diagrams at all.
Ex: String theory and $\mathcal{N} = 4$ SYM.

Superenormalisable: Finite number of infinite diagrams, $[\lambda] = L^{<0}$.
Ex: $[m^2] = L^{-2}$ and ϕ^3 in $d = 4$.
(Such interactions called *relevant* in condensed matter physics.)

Renormalisable: Infinite number of infinite diagrams but only for small N, $[\lambda] = L^{\leq 0}$.
Ex: ϕ^4 in $d = 4$.
(Called *marginal* in condensed matter.)

Non-renormalisable: Infinite number of infinite diagrams at all values of N, $[\lambda] = L^{>0}$.
Ex: ϕ^5 in $d = 4$ and gravity with $[G] = L^2$. (Called *irrelevant* in condensed matter.)

[3]This case is relevant in M-theory, the theory that unifies all string theories.

To make these ideas concrete let us return to ϕ^4 in $d = 4$ theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{1.97}$$

This theory has three quantities

$$[\phi] = L^{-1}, \quad [m] = L^{-1}, \quad [\lambda] = L^0, \tag{1.98}$$

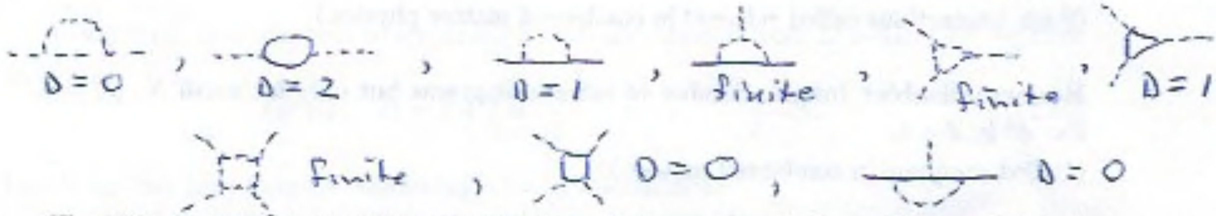
that are associated with the renormalisation constants δ_Z, δ_m and δ_λ which can be used to cancel infinities arising in the perturbation expansion. The number of places infinities can occur must therefore not exceed three in a sense that will become clear below: $(\phi^4)|_4$ has

$$D = 4 - N \Rightarrow \begin{cases} N = 2 : D = 2 \to \Lambda^2 \Rightarrow 2 \text{ infinities} \\ N = 4 : D = 0 \to \log\Lambda \Rightarrow 1 \text{ infinity} \\ N = 6 : D = -2 \to \Lambda^{-2} \Rightarrow \text{no infinities.} \end{cases} \tag{1.99}$$

Here we need to recall the argument (expansion the propagator in powers of p^2) that led to the conclusion that $D = 2$ corresponds to two kinds of infinities. The conclusion from this counting exercise is that the number of infinities is 3 which equals the number of renormalisation constants in the theory making it *renormalisable*. These infinities also arise in the right places for this conclusion to be valid.

Comment: As we will see later this counting of infinities and renormalisation constants becomes more involved but also much more interesting when symmetries enter the situation, either gauge symmetries as in QED (see below) or global ones as for instance $\phi \to -\phi$ which is present in the ϕ^4 in $d = 4$ theory discussed here.

To exemplify this comment we consider instead Yukawa theory with a Dirac fermion coupled via $g\phi\bar{\psi}\psi$ to a real scalar with self-interaction $\lambda_3\phi^3$ in $d = 4$. This theory is renormalisable according to the classification above since g is dimensionless and λ_3 has dimension L^{-1}. But the scalar potential is potentially bad since it is unbounded from below and the theory is therefore unstable. However, let us consider the perturbation theory despite this fact. There are a number of divergent graphs, the bosonic snail, the fermionic snail, the fermi triangle, the fermi square etc: (solid line=fermion, dashed line =scalar)



Thus the theory is not renormalisable since $\mathcal{L}$ does not contain a ϕ^4 term which is generated in perturbation theory by a diagram that is infinite. The conclusion is that we have to add the ϕ^4 term to the Lagrangian which then becomes both renormalisable and stable. The same argument would have applied also to the ϕ^3 term if it had not been present in the Lagrangian from the start.

Exercise: Verify that this Yukawa theory with both cubic and quartic scalar interaction terms is renormalisable by counting divergencies and renormalisation constants.

The above conclusion about the need to add the quartic term, as well as the cubic one, is very general and can be expressed as follows:

Rule for renormalisability: All possible renormalisable terms must be included in the Lagrangian unless they are forbidden by symmetries. Each renormalisation parameter δ requires an experiment to determine the related physical parameter in the Lagrangian.

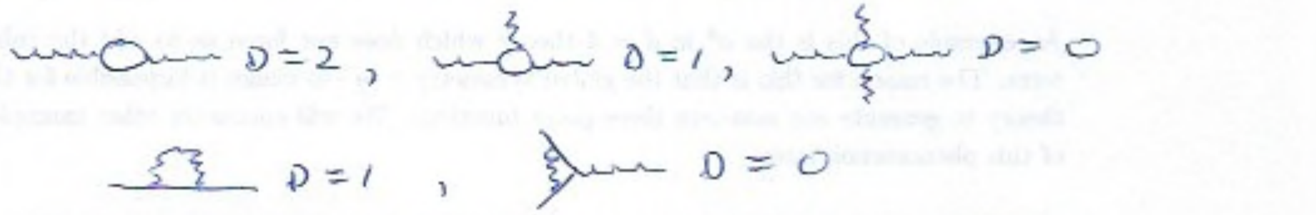
An example of this is the ϕ^4 in $d = 4$ theory which does not force us to add the cubic term. The reason for this is that the global symmetry $\phi \to -\phi$ makes it impossible for the theory to generate any non-zero three-point functions. We will encounter other examples of this phenomenon later.

1.2 Renormalisation of QED (in $d = 4$)

It is interesting that we can apply renormalised perturbation theory (i.e., based on a Lagrangian with counter terms) to QED even before we have computed any of its divergent diagrams. The Lagrangian is (before renormalisation)

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi. \tag{1.100}$$

This theory has two fields, A_μ and ψ, and two parameters m and e, and hence four renormalisation constants (the δ-parameters in the renormalised Lagrangian). Thus there can not be more than four divergencies in perturbation theory, and they must appear in the right diagrams. Let us check if this is the case: The divergent diagrams are[4]



Thus we have found 7 infinities! Note that get this number both the photon propagator with $D = 2$ and the Dirac propagator with $D = 1$ correspond to two divergences each. This can be seen by expanding these propagators, in p^2 and $\not{p}$, respectively.

These values of D follow also from the QED formula (N_ψ is always even ($=N_e$ in PS))

$$D(QED) = 4 - N_\gamma - \frac{3}{2}N_\psi, \tag{1.101}$$

which can be derived in a similar fashion to the formula for scalar fields above. The above divergent diagrams can then be extended to the whole perturbation series and written as blob-diagrams.

This QED result requires some deeper analysis to see if the theory can be rescued.

Let us start this analysis by looking at the photon two-point function with $D = 2$:

$$\Pi^{\mu\nu}(q^\mu) = g^{\mu\nu}A(q^2) + q^\mu q^\nu B(q^2), \tag{1.102}$$

for some functions $A(q^2)$ and $B(q^2)$. Note that $\Pi^{\mu\nu}(q^\mu)$ is symmetric in its two indices because the external particles are identical bosons (the same is true for other photon vertex functions given above). But this amplitude must satisfy the Ward identity applied to any one of the two photon legs. Thus

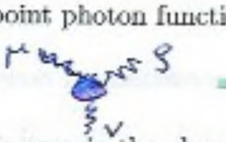
$$q_\mu\Pi^{\mu\nu} = 0 \Rightarrow \Pi^{\mu\nu} = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2). \tag{1.103}$$

[4]Also the vacuum diagram is infinite but we may consider it irrelevant since it was cancelled in the previous discussion of correlation functions and their expansions in terms of connected graphs only.

This does indeed improve the situation since this condition eliminates one of the two divergencies in $\Pi^{\mu\nu}$: $\Pi(q^2)$ has $D = 0$.
Note that since we are here dealing with the 1PI two-point function there is no $1/q^2$ in $\Pi(q^2)$ that can eliminate the q^2 in the metric term. The divergence that remains is $\log \Lambda$.

Turning to the four-point function with four external photon legs, it must also satisfy the Ward identity on any of its legs. The amplitude $\mathcal{M}^{\mu\nu\rho\sigma}(k_1, k_2, k_3, k_4)$ is a rather complicated expression (but is symmetric in exchange of any two external legs) but the way to satisfy the Ward identity is to note that a factor $g^{\mu\nu}k_1^\rho - g^{\mu\rho}k_1^\nu$ satisfies $q_\mu(g^{\mu\nu}k_1^\rho - g^{\mu\rho}k_1^\nu) = 0$ that is the Ward identity for the first photon leg (associated to the index μ and the momentum k_1). This idea can be applied to each leg which then solves the problem and gives the whole $\mathcal{M}^{\mu\nu\rho\sigma}(k_1, k_2, k_3, k_4)$ as four such factors times a function $\mathcal{M}(k_1, k_2, k_3, k_4)$ with $D = -4$. This eliminates one more divergence.

Fortunately, there is one more feature related to gauge invariance that can be used. The exact three-point photon function can be expressed as follows



$$= \langle \Omega | j^\mu j^\nu j^\rho | \Omega \rangle = 0. \tag{1.104}$$

The reason this is zero is the charge conservation of the ground state $C|\Omega\rangle = |\Omega\rangle$ and $Cj^\mu C^{-1} = -j^\mu$. Incidentally, this argument also tells us that the photon one-point function vanishes which is good news since non-zero one-point functions indicate that the theory is unstable (recall home problem 1 before the VEV v was given its value at the bottom of the mexican hat potential). Both of these conclusions are valid beyond perturbation theory as the notation indicates.

There are two more interesting comments to make here. The full QED vertex (here ab are spinor indices which most often are not written out)

$$(\Gamma^\mu)_a{}^b = (\gamma^\mu)_a{}^b + ..., \Rightarrow D = 0, \tag{1.105}$$

is also $\log \Lambda$ divergent. The final comment concerns the full Dirac propagator

$$\Sigma_a{}^b = (S_F)_a{}^b + ... \Rightarrow D = 1, \tag{1.106}$$

and therefore has two divergencies since, at the subtraction point $\not{p} = m$ it can be expanded in $\not{p}$ as $a_0\Lambda + a_1\not{p}\log\Lambda$. However, if the fermion is massless ($m = 0$) then the theory is chiral and no mass term can be generated in perturbation theory since there are then no Feynman rules mixing ψ_L and ψ_R. Hence the $\not{p}$ independent divergent term (as Λ) must be proportional to m. We conclude therefore that for dimensional reasons the first term is only $\log \Lambda$ divergent:

$$\Sigma_a{}^b \propto a_0 m \log\Lambda\, \delta_a{}^b + a_1 \log\Lambda\, (\not{p})_a{}^b. \tag{1.107}$$

It is rather nice that gauge invariance eliminates precisely three divergencies which is the required number to make QED renormalisable. Thus, using also the last chirality argument, we find that all four remaining divergencies are of the type $\log \Lambda$.

We can now continue the process of constructing the renormalised Lagrangian for QED. The conventional multiplicative renormalisation of the fields read

$$\psi := \sqrt{Z_2}\psi^r, \quad A_\mu := \sqrt{Z_3}A^r_\mu. \tag{1.108}$$

Then starting from the above QED Lagrangian but now with an index 0 to indicate the bare constants, it reads in terms of the renormalised fields just defined:

$$\mathcal{L} = -\frac{1}{4}Z_3(F^r_{\mu\nu})^2 + Z_2\bar{\psi}^r(i\gamma^\mu\partial_\mu - m_0)\psi^r - e_0 Z_2\sqrt{Z_3}A^r_\mu\bar{\psi}^r\gamma^\mu\psi^r. \tag{1.109}$$

It is now very important that if we use a regularisation procedure that respects gauge invariance, like dimensional regularisation or Pauli-Villars, then the last two terms must combine to a covariant derivative $D_\mu = \partial_\mu - ieA_\mu$ where e must be the physical coupling constant. This implies that (in perturbation theory as well as exactly)

$$e = e_0\sqrt{Z_3}. \tag{1.110}$$

It is conventional to define another multiplicative renormalisation constant Z_1 from the interaction term by

$$eZ_1 := e_0 Z_2\sqrt{Z_3}. \tag{1.111}$$

The relation derived from gauge invariance in the previous equation then implies

$$Z_1 = Z_2. \tag{1.112}$$

We will come back to this condition later and show that it is correct at least to first loop order in perturbation theory provided the regularisation method we use respects gauge invariance (i.e., dimensional and Pauli-Villars).

We can now take the second standard step and define the renormalised QED Lagrangian. This is done by defining the δ-parameters

$$\delta_1 := Z_1 - 1, \quad \delta_2 := Z_2 - 1, \quad \delta_3 := Z_3 - 1, \quad \delta_m := Z_2 m_0 - m. \tag{1.113}$$

With these definitions the renormalised Lagrangian becomes

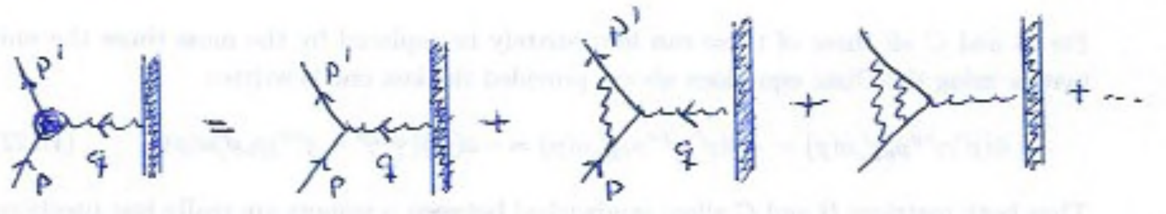
$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}(F^r_{\mu\nu})^2 + \bar{\psi}^r(i\gamma^\mu\partial_\mu - m)\psi^r - eA^r_\mu\bar{\psi}^r\gamma^\mu\psi^r \\ &-\frac{1}{4}\delta_3(F^r_{\mu\nu})^2 + \bar{\psi}^r(i\delta_2\gamma^\mu\partial_\mu - \delta_m)\psi^r - e\delta_1 A^r_\mu\bar{\psi}^r\gamma^\mu\psi^r.\end{aligned} \tag{1.114}$$

We will now start computing the expressions for the divergent one-loop diagrams that we have identified above, and also show that no other divergences arise that would contradict the conclusions found here. These divergent Feynman graphs appear in the vertex and in the two propagators, i.e., in the quantities

$$(\Gamma^\mu)_a{}^b, \quad \Pi^{\mu\nu}, \quad \Sigma_a{}^b. \tag{1.115}$$

1.2.1 Renormalised QED: the vertex $(\Gamma^\mu)_a{}^b$

To discuss the general structure of the vertex function $\Gamma^\mu(p', p)$ we may consider it as part of a scattering process of an electron hitting a fixed (very heavy) target, e.g., a nucleus. We draw this as



which gives rise to the following expression for the scattering amplitude, where $q = p' - p$,

$$i\mathcal{M} = \left(\bar{u}(p')(-ie\Gamma^\mu)u(p)\right)\frac{-ig_{\mu\nu}}{q^2+i\epsilon}\left(u(k')(-ie\gamma^\nu)u(k)\right) = ie^2\frac{\left(\bar{u}(p')\Gamma^\mu u(p)\right)\left(u(k')\gamma_\mu u(k)\right)}{q^2+i\epsilon}. \tag{1.116}$$

Here we have approximated the nucleus by the first perturbative term $u(k')\gamma_\mu u(k)$. Clearly, to first order in perturbation theory the vertex function is just the γ^μ from the vertex Feynman rule, i.e.,

$$\Gamma^\mu(p, p')|_{1st\ order} = \gamma^\mu. \tag{1.117}$$

Below we will compute the first loop correction of the vertex function $\Gamma^\mu(p, p')$.

In general, the full expression for the vertex function $\Gamma^\mu(p, p')$ (from diagrams at all orders in perturbation theory) can only depend on

$$p'^\mu, p^\mu, \gamma^\mu, e, m. \quad (q^\mu = p'^\mu - p^\mu). \tag{1.118}$$

With this input we can try to determine the expression for $\Gamma^\mu(p, p')$ as much as possible from general properties that it has to satisfy.

We first use Lorentz invariance. It implies that

$$\Gamma^\mu(p, p') = \gamma^\mu A + (p'^\mu + p^\mu)B + (p'^\mu - p^\mu)C, \tag{1.119}$$

where A, B, C are three independent 4×4 bi-spinorial matrices which are functions of p'^μ and p^μ, and which have no vector indices. These matrices A, B, C can be expanded in the γ-basis $\{\gamma^{(n)}\}$ with the elements contracted into momenta p'^μ and p^μ. For example, they may contain a term $f(p, p')\gamma_{\mu\nu}p^\mu p'^\nu$ while $f(p, p')\gamma_{\mu\nu}p^\mu p^\nu = 0$. Also, no non-zero factors with $\gamma_{\mu\nu\rho}$ and $\gamma_{\mu\nu\rho\sigma}$ can exist (since there are only two independent momenta in the vertex).

At this point we can use the fact that Γ^μ is sandwiched between $u(p)$ and $\bar{u}(p')$ spinors in $\mathcal{M}$ which satisfy the Dirac equations:

$$\not{p}\, u(p) = m\, u(p), \quad \bar{u}(p')\, \not{p}' = m\, \bar{u}(p'). \tag{1.120}$$

It is then possible to simplify the matrix functions A, B, C (functions of p and p') quite a bit. In fact, recalling the comments above, the only kinds of non-trivial matrix terms (i.e., not proportional to the unit matrix) are

$$\gamma^\nu p_\nu,\ \gamma^\nu p'_\nu,\ \gamma^{\nu\rho} p_\nu p'_\rho. \tag{1.121}$$

For B and C all three of these can immediately be replaced by the mass times the unit matrix using the Dirac equations above, provided the last one is written

$$\bar{u}(p')\gamma^{\nu\rho} p_\nu p'_\rho u(p) = -\bar{u}(p')\gamma^{\rho\nu} p_\nu p'_\rho u(p) = -\bar{u}(p')(\gamma^\rho\gamma^\nu - g^{\nu\rho}) p_\nu p'_\rho u(p). \tag{1.122}$$

Thus both matrices B and C when sandwiched between u spinors are really just functions of the momenta times the unit matrix. The same conclusion is true for the matrix function A but here one has to use also

$$\bar{u}(p')\,\gamma^\mu \not{p}' = \bar{u}(p')\,\gamma^\mu\gamma^\nu p'_\nu = \bar{u}(p')\,(-\gamma^\nu\gamma^\mu p'_\nu + 2p'^\mu) = -\bar{u}(p')\,\not{p}' + 2\bar{u}(p')\,p'^\mu. \tag{1.123}$$

To summarise: All three matrices A, B, C are just Lorentz invariant functions of the momenta p and p' (since $q = p' - p$) times the unit matrix. There are only three Lorentz invariants in this case, p^2, p'^2 and $p \cdot p'$, but on-shell we have $p^2 = p'^2 = m^2$. We can also use $q^2 = (p' - p)^2 = 2m^2 - 2p \cdot p'$ to express $p \cdot p'$ in terms of q^2. Thus we conclude that the A, B, C are functions of just q^2, and the parameters m and e.

As a second constraint on the vertex function it must of course satisfy the Ward identity. The photon leg has momentum $q = p' - p$ so the Ward identity

$$q_\mu \Gamma^\mu(p, p') = 0 \Rightarrow (p'_\mu - p_\mu)\gamma^\mu A(q^2) + (p'_\mu - p_\mu)(p'^\mu + p^\mu)B(q^2) + q^2 C(q^2) = 0, \tag{1.124}$$

where it is important to emphasise that in the Ward identity the photon momentum q^μ is arbitrary and does not have to satisfy $q^2 = 0$.

When sandwiching the above expression between u spinors we can as above use the Dirac equation in both directions (i.e., on both $u(p)$ and $\bar{u}(p')$) which sets the first term to zero. Also the second term is zero using $p^2 = p'^2 = m^2$. Thus it follows that $C(q^2) = 0$ and we have shown that

$$\Gamma^\mu(p, p') = \gamma^\mu A(q^2) + (p'^\mu + p^\mu)B(q^2), \tag{1.125}$$

where both $A(q^2)$ and $B(q^2)$ are just functions times the unit matrix. This result is valid beyond perturbation theory and turns out to have a lot of physics in it.

To get a clear picture of the physics it turns out convenient to replace the second term containing a factor $(p'^\mu + p^\mu)$ with a term involving $\sigma^{\mu\nu} q_\nu$. This can be done using the so called **Gordon identity**:

$$\bar{u}(p')\,\gamma^\mu\, u(p) = \bar{u}(p')\left(\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu} q_\nu}{2m}\right) u(p). \tag{1.126}$$

The proof of this identity goes as follows: First note that

$$i\sigma^{\mu\nu}q_\nu = -\gamma^{\mu\nu}q_\nu = -\frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(p'_\nu - p_\nu) = -\frac{1}{2}(\gamma^\mu(\not{p}' - \not{p}) - (\not{p}' - \not{p})\gamma^\mu). \quad (1.127)$$

When sandwiched between u spinors this becomes (using $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$)

$$\bar{u}(p')i\sigma^{\mu\nu}q_\nu u(p) = -\frac{1}{2}\bar{u}(p')(\gamma^\mu\not{p}' - 2m\gamma^\mu + \not{p}\gamma^\mu)u(p) = \bar{u}(p')(2m\gamma^\mu - (p'^\mu + p^\mu))u(p). \quad (1.128)$$

So using the Gordon identity to eliminate $(p'^\mu + p^\mu)$ we get

$$\Gamma^\mu(p,p') = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2), \quad (1.129)$$

where the functions $F_i(q^2)$ are called **form factors**. As we will now show the two form factors have the following interpretation

$F_1(q^2)$: Electric properties of the electron interactions.
Example: as when coupled to a classical potential from a fixed target $A_\mu^{class}(x) = (\phi^{class}(\mathbf{r}), 0, 0, 0)$.
$F_2(q^2)$ together with $F_1(q^2)$: Magnetic properties of the electron interactions.
Example: as when coupled to $A_\mu^{class}(x) = (0, \mathbf{A}^{class}(\mathbf{r}))$ from a fixed target.

Before we explain the connection to physics we note that the vertex function $\Gamma^\mu(p,p')$, considered as the exact expression (i.e., the complete result to all orders in perturbation theory) is related to how we measure the electric charge e. Thus it is natural to set the subtraction point for the charge in renormalised QED at $q^\mu = 0$ in the same way as we did for the coupling constant in ϕ^4 previously (there we used $\mathbf{p_1} = \mathbf{p_2} = 0$ which implied $s = 4m^2$ and $t = u = 0$ in the four-point amplitude). Thus the renormalisation condition for the charge e in QED, at the subtraction point $q^\mu = 0$, is

$$\Gamma^\mu(p,p')|_{q^\mu=0} = \gamma^\mu, \quad (1.130)$$

which corresponds to $F_1(q^2 = 0) = 1$.

Electric case: Since the classical potential from a fixed source is time independent its Fourier transform becomes (dropping *class* on the RHS)

$$\tilde{A}_\mu^{class}(q) = 2\pi\delta(q^0)(\tilde{\phi}(\mathbf{q}), 0, 0, 0). \quad (1.131)$$

The scattering amplitude then reads

$$i\mathcal{M} = -ie\bar{u}(p')\Gamma^0(p',p)u(p)\tilde{\phi}(\mathbf{q})|_{q^0=0}. \quad (1.132)$$

Then if we assume that the potential is roughly constant over a large volume of space we have also $\tilde{\phi}(\mathbf{q})|_{\mathbf{q}\approx 0}$. Therefore, in the limit $q^\mu \to 0$ we have

$$\Gamma^\mu(p,p')|_{q^0=0,\,\mathbf{q}\to 0} = \gamma^\mu F_1(0). \quad (1.133)$$

For slowly moving electrons hitting the static potential of the nucleus we can use the non-relativistic approximation

$$\bar{u}(p')\gamma^0 u(p) = u^\dagger(p')u(p) \approx 2m\xi'^\dagger\xi. \tag{1.134}$$

Thus

$$i\mathcal{M} = -ieF_1(0)\tilde{\phi}(\mathbf{q})2m\xi'^\dagger\xi|_{\mathbf{q}\to 0}. \tag{1.135}$$

Finally, this result should be compared to the Born approximation $\langle p|iT|p'\rangle = -i\tilde{V}(\mathbf{q})$ for the potential

$$V(\mathbf{r}) = e\phi(\mathbf{r})F_1(0), \tag{1.136}$$

which gives the same answer as QED for a fixed potential. Here we can use the fact that e is the charge at the subtraction point so in fact $F_1(0) = 1$.

Magnetic case: Here we are interested in understanding the physics of F_2. Consider therefore (again dropping *class* on the RHS)

$$A_\mu^{class}(x) = (0, \mathbf{A}(\mathbf{r})). \tag{1.137}$$

Since we in this case want to express the scattering amplitude in terms of the magnetic field $\mathbf{B}(\mathbf{r})$ coupled to the spin $\mathbf{S}$, we need to expand the quantities to linear order in the momentum q^μ. To get the physics it is sufficient to use the non-relativistic approximation of the spinors:

$$u_L(p) = \sqrt{p\cdot\sigma}\xi \approx \sqrt{E}(1 - \frac{p^i\sigma^i}{2E})\xi, \quad u_R(p) = \sqrt{p\cdot\bar{\sigma}}\xi \approx \sqrt{E}(1 + \frac{p^i\sigma^i}{2E})\xi, \tag{1.138}$$

and similarly for $\bar{u}(p')$. This gives, with $E \approx m$,

$$\bar{u}(p')\gamma^i u(p) = 2m\xi'^\dagger \left(\frac{\mathbf{p}'\cdot\sigma}{2m}\sigma^i + \sigma^i\frac{\mathbf{p}\cdot\sigma}{2m}\right)\xi. \tag{1.139}$$

Picking up the σ^i terms and inserting them into the expression for the scattering amplitude gives

$$i\mathcal{M} = -i2me\xi'^\dagger\left(-\frac{1}{2m}\sigma^i(F_1(0)+F_2(0))\right)\xi\tilde{B}^i(\mathbf{q})), \tag{1.140}$$

where we obtained the magnetic field in the form

$$\tilde{B}^i(\mathbf{q}) = -i\epsilon^{ijk}q^j\tilde{A}^k_{class}(\mathbf{q}). \tag{1.141}$$

Comparing to the Born approximation again we find that it comes from the **magnetic moment** interaction with potential energy

$$V(\mathbf{r}) = -\langle\boldsymbol{\mu}\rangle\cdot\mathbf{B}(\mathbf{r}), \tag{1.142}$$

where

$$\langle\boldsymbol{\mu}\rangle = \frac{e}{m}(F_1(0)+F_2(0))\xi'^\dagger\frac{\boldsymbol{\sigma}}{2}\xi. \tag{1.143}$$

From this expression we can determine the Lande's g-factor from its definition

$$\boldsymbol{\mu} = g(\frac{e}{2m})\boldsymbol{S}. \tag{1.144}$$

Thus we find the g-factor to be given by

$$g = 2(F_1(0) + F_2(0)) = 2 + 2F_2(0), \tag{1.145}$$

where we have used the fact that $F_1(0) = 1$ at the subtraction point $q^\mu = 0$ in renormalised QED. This result is often presented as an expression for $g - 2$ which is the so called **anomalous** magnetic moment, i.e.,

$$g - 2 = 2F_2(0). \tag{1.146}$$

The RHS starts at one loop order, i.e., at $\mathcal{O}(\alpha)$, but can be computed to arbitrary order in the fine structure constant and we will obtain the first term below. The computation of $g-2$ and the comparison to experiments is often quoted as one of the most accurate results in natural science and the discrepancy between theory and the measured value arises at the 11th or 12th decimal point. This is discussed by David Gross at 2011 Solvay conference which you are strongly recommended to have a look at as part of the course. To indicate the complexity of this calculation there are about 1000 Feynman diagrams at the four-loop order and 12.672 diagrams at the 5-loop order[5].

Comment: The expression above defining the form factors F_1 and F_2 can in fact be extended if parity invariance and time-reversal symmetry are not assumed to hold. This situation is the one encountered for the weak nuclear forces with gauge group $SU(2)$ in the standard model so this might be a relevant thing to do in a context that is more general than just QED. So dropping this assumption we can also use tensors that break parity and time-reversal symmetry, that is $\epsilon^{\mu\nu\rho\sigma}$ and γ^5. This leads to

$$\Gamma^\mu(p,p') = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2) + i\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}q_\nu F_3(q^2) + \frac{1}{2m}(q^\mu - \frac{q^2}{2m}\gamma^\mu)\gamma^5 F_4(q^2). \tag{1.147}$$

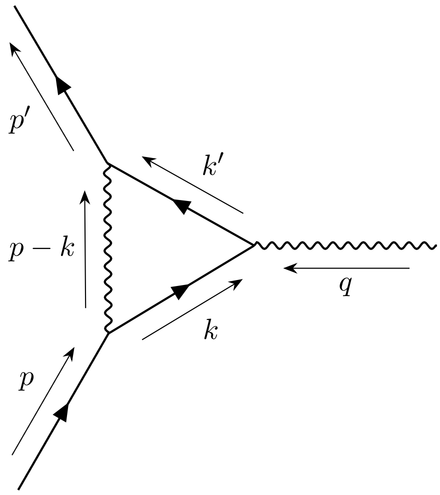
In particular, F_3, the **electric dipole moment**, is of importance since if non-zero it implies CP violation which is required in the study of the early universe to explain the matter-antimatter asymmetry that we observe in the universe today[6].

[5] See hep-th 1208.6583 and CERN news April 2017.

[6] See Eckel et al, physics.atom-ph/1208.4420 (in Phys Rev Letters 109 (2012) 193003).

The vertex at one loop

We now turn to the actual evaluation of the vertex function at first loop order. The diagram is



$$(1.148)$$

which generates the following expression for the one-loop correction $\delta\Gamma^\mu(p,p')$ to the tree-result $\Gamma^\mu(p,p')|_{tree} = \gamma^\mu$ (with $k' = k+q$):

$$\delta\Gamma^\mu(p,p') = \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\nu\rho}}{(p-k)^2+i\epsilon} \bar{u}(p')(-ie\gamma^\nu) \frac{i(\not{k}'+m)}{k'^2-m^2+i\epsilon} (\gamma^\mu) \frac{i(\not{k}+m)}{k^2-m^2+i\epsilon} (-ie\gamma^\rho) u(p) \tag{1.149}$$

$$= 2ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p')\left(\not{k}\gamma^\mu\not{k}' + m^2\gamma^\mu - 2m(k^\mu + k'^\mu)\right)u(p)}{((p-k)^2+i\epsilon)(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}. \tag{1.150}$$

Note that the first vertex is contracted into the third by the metric from the photon propagator in the loop. This makes it possible to use the identities (derived in previous lectures) $\gamma^\nu\gamma^\mu\gamma_\nu = -2\gamma^\mu$, $\gamma^\rho\gamma^{\mu\nu}\gamma_\rho = 0$ and $\gamma^\nu\not{k}'\gamma^\mu\not{k}\gamma_\nu = -2\not{k}\gamma^\mu\not{k}'$.

Doing this integral is not easy but we have already used the methods needed in the simpler context of the ϕ^4 theory. The steps used there can be applied again although this time they will require some generalisation. First, we note that the vertex integral has three propagators and we have only consider loops with two propagators before. This means that we must derive a version of the Feynman parameter trick that can cope with three propagators. Recall the formula we used previously with one Feynman parameter x:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA+(1-x)B)^2} = \int_0^1 dx \int_0^1 dy\, \delta(x+y-1) \frac{1}{(xA+yB)^2}, \tag{1.151}$$

where we have introduced a second parameter y in a trivial manner since it can be eliminated directly using the δ-function. However, this second version has a generalisation to any number n of Feynman parameters:

$$\frac{1}{A_1...A_n} = \int_0^1 dx_1...dx_n\, \delta(x_1+....x_n-1) \frac{(n-1)!}{(x_1A_1+....+x_nA_n)^n}. \tag{1.152}$$

The proof is a bit messy but is most easily done using induction. The proof is not that important for the rest of the course but here it is:
It is obviously true for $n = 2$ from above so we should assume it is true for some arbitrary n and prove that it is also true for $n + 1$. The first step is to multiply the above general formula by $\frac{1}{A}$ which gives (renaming the the x_i-parameters $\bar{x}_i$)

$$\frac{1}{AA_1...A_n} = \int_0^1 d\bar{x}_1...d\bar{x}_n\,\delta(\Sigma_i\bar{x}_i - 1)\,\frac{(n-1)!}{A(\Sigma_i\bar{x}_iA_i)^n}, \tag{1.153}$$

where all sums run over $i = 1, 2, ..., n$.

Then second step is to apply $n-1$ B-derivatives $(\frac{\partial}{\partial B})^{n-1}$ to the above equation for $\frac{1}{AB}$. This gives

$$\frac{1}{AB^n} = \int_0^1 dx \int_0^1 dy\,\delta(x+y-1)\frac{ny^{n-1}}{(xA+yB)^{n+1}}. \tag{1.154}$$

The by setting $B = \Sigma_i\bar{x}_iA_i$ we get the integrand in the previous formula which can thus be written as

$$\frac{1}{AA_1...A_n} = \int_0^1 dx \int_0^1 dy\,\delta(x+y-1) \int_0^1 d\bar{x}_1...d\bar{x}_n\,\delta(\Sigma_i\bar{x}_i-1)\,\frac{n!y^{n-1}}{(xA+y\Sigma_i\bar{x}_iA_i)^{n+1}}. \tag{1.155}$$

The final step is to use the $\delta(x + y - 1)$ to do the y-integral which gives $y = 1 - x$ and hence

$$\frac{1}{AA_1...A_n} = \int_0^1 dxd\bar{x}_1...d\bar{x}_n\,\delta(\Sigma_i\bar{x}_i - 1)\,\frac{n!(1-x)^{n-1}}{(xA+(1-x)\Sigma_i\bar{x}_iA_i)^{n+1}}. \tag{1.156}$$

That this is the correct result which proves the induction step becomes clear by the substitution $x_i = (1-x)\bar{x}_i$, and noting that the measure becomes

$$d^n\bar{x}\,\delta(\Sigma_i\bar{x}_i - 1)(1-x)^{n-1} = d^nx\,\delta(\frac{\Sigma_ix_i}{1-x} - 1)\frac{1}{1-x} = d^nx\,\delta(\Sigma_ix_i - (1-x)) = d^nx\delta(x + \Sigma_ix_i - 1). \tag{1.157}$$

Thus we get

$$\frac{1}{A_1...A_{n+1}} = \int_0^1 dx_1...dx_{n+1}\,\delta(x_1 + ...x_{n+1} - 1)\,\frac{n!}{(x_1A_1 + ... + x_{n+1}A_{n+1})^{n+1}}, \tag{1.158}$$

which is the same formula as above but now for $n + 1$ factors.

The version we need in the vertex integral is the one with three parameters x, y, z. We then get the propagator factors in the form

$$\frac{1}{((k-p)^2 + i\epsilon)(k'^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)} = \int_0^1 dxdydz\,\delta(x+y+z-1)\frac{2}{D^3}, \tag{1.159}$$

where

$$D = x(k^2 - m^2) + y(k'^2 - m^2) + z((k-p)^2) + (x+y+z)i\epsilon. \tag{1.160}$$

If we use the fact that $x+y+z=1$ and perform a shift of the momentum k to $l := k+yq-zp$ this expression simplifies to

$$D = l^2 - \Delta + i\epsilon, \quad \text{where} \ \ \Delta := -xyq^2 + (1-z)m^2. \tag{1.161}$$

Note that since $q^2 < 0$ we see that $\Delta > 0$ and hence Δ can be regarded as an addition to the mass (in the second term).

Next we turn to the expression in the numerator of the integral in the vertex function. Since the denominator after the momentum shift is an even function of l (it depends only on l^2) the whole integrand will contain an odd part proportional to l^μ and an even part proportional to $l^\mu l^\nu$:

$$\int \frac{d^4l}{(2\pi)^4}\frac{l^\mu}{D^3} = 0, \quad \int \frac{d^4l}{(2\pi)^4}\frac{l^\mu l^\nu}{D^3} = \int \frac{d^4l}{(2\pi)^4}\frac{\frac{1}{4}g^{\mu\nu}l^2}{D^3}. \tag{1.162}$$

The first result follows directly since the integrand is an odd function of each component of l^μ. The second one is true since when integrating over the angles in momentum space one obtains a covariant direction independent result which can be expressed as the RHS (one can contract the indices to check the coefficient).

The numerator can be simplified further by using the manipulations (Dirac equation etc) used above to arrive at the general vertex expression

$$\Gamma^\mu(p,p') = \gamma^\mu\, F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\, F_2(q^2). \tag{1.163}$$

Doing this we find (after some work) that the numerator is

$$\begin{aligned}&\bar{u}(p')\left(\gamma^\mu(-\frac{1}{2}l^2 + (1-x)(1-y)q^2 + (1-2z-z^2)m^2) + (p'^\mu + p^\mu)mz(z-1)\right)u(p)\\ &+\bar{u}(p')(q^\mu m(z-2)(x-y))u(p).\end{aligned} \tag{1.164}$$

We have written the last term separately since we know from the application of the Ward identity that a term proportional to q^μ (without $\sigma^{\mu\nu}$) must vanish. Here we can check that fact: The integral over the Feynman parameters x and y is symmetric in $x \leftrightarrow y$ while the last term above is odd and hence vanishes.

To get this expression into a form useful for the physics interpretation we use again the Gordon identity. This gives the final version of the vertex correction at one loop:

$$\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) = 2ie^2\int \frac{d^4l}{(2\pi)^4}\int_0^1 dxdydz\delta(x+y+z-1)\frac{2}{D^3}$$

$$\times\bar{u}(p')\left(\gamma^\mu(-\frac{1}{2}l^2 + (1-x)(1-y)q^2 + (1-4z-z^2)m^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}(2m^2z(1-z))\right)u(p), \tag{1.165}$$

where

$$D = l^2 - \Delta + i\epsilon, \quad \Delta = -xyq^2 + (1-z)^2 m^2 > 0. \tag{1.166}$$

Looking at this result we can start identifying the contributions to the form factors F_1 and F_2.

At this point we can start using techniques that we developed in the context of ϕ^4 theory: Wick rotation and the three kinds of regularisation method. The reason we need to regularise is, as we have already noticed, the vertex function is $\log \Lambda$ (cut-off) divergent. Performing the Wick rotation as usual by setting

$$l^0 = i l_E^0, \quad l^2 = -l_E^2, \tag{1.167}$$

and using our previously obtained result for the angular integrals in $d = 4$, namely $\int d\Omega_4 = 2\pi^2$, we get for the two integrals that appear in the expression:

$$\int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - \Delta)^3} \to \frac{-i2\pi^2}{(2\pi)^4} \int_0^\infty dl_E \frac{l_E^3}{(l_E^2 + \Delta)^3}. \tag{1.168}$$

which is finite, while the following one is infinite

$$\int \frac{d^4 l}{(2\pi)^4} \frac{l^2}{(l^2 - \Delta)^3} \to \frac{i2\pi^2}{(2\pi)^4} \int_0^\infty dl_E \frac{l_E^5}{(l_E^2 + \Delta)^3}. \tag{1.169}$$

We choose here (following PS) to apply the method of Pauli-Villars to the photon propagator in the loop integral. Thus

$$\frac{1}{(k-p)^2 + i\epsilon} \to \frac{1}{(k-p)^2 + i\epsilon} - \frac{1}{(k-p)^2 - \Lambda^2 + i\epsilon}, \tag{1.170}$$

where we have introduced a heavy ghost photon with mass Λ (again following PS) with a minus sign in front of its propagator. These two terms then cancel each other for very large momenta k turning the integral UV finite.

Finally, we obtain the following results for the renormalised form factors at order $\mathcal{O}(\alpha)$. After an infinite subtraction at $q^\mu = 0$ in the case of F_1, it reads[7]

$$F_1(q^2) = 1 + \frac{\alpha}{2\pi} \int_0^1 dxdydz\delta(x+y+z-1)$$

$$\times \left(\log\left(\frac{m^2(1-z)^2}{m^2(1-z)^2 - q^2xy}\right) + \frac{m^2(1-4z+z^2) + q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2xy} - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2 - q^2xy} \right), \tag{1.171}$$

where we see explicitly that setting $q^\mu = 0$ reduces this result to $F_1(0) = 1$. In the case of F_2 no subtraction in needed so the result is given directly by the QED calculation which gives

$$F_2(q^2) = \frac{\alpha}{2\pi} \int_0^1 dxdydz\delta(x+y+z-1) \left(\frac{2m^2 z(1-z)}{m^2(1-z)^2 - q^2xy} \right). \tag{1.172}$$

[7] The small photon ghost mass μ that is used in PS to make the integral regular in the IR is not relevant for our discussion and has therefore been set to zero here.

In view of the fact that F_2 is not involved in the renormalisation it is reassuring that the final answer in this case is finite at both UV and IR. We are interested in comparing this result to experiments at low energies so we need

$$F_2(q^2 = 0) = \frac{\alpha}{2\pi} \int_0^1 dx dy dz \delta(x + y + z - 1) \left(\frac{2m^2 z(1 - z)}{m^2(1 - z)^2} \right). \tag{1.173}$$

The integrals over the Feynman parameters can now be done (over a flat triangular surface between the three points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$): Simplifying the integrand and using the δ-function to do the x integral reduces the integral to a triangle in the yz-plane:

$$F_2(q^2 = 0) = \frac{\alpha}{\pi} \int_0^1 dz \int_0^{1-z} dy \left(\frac{z}{1 - z)} \right) = \frac{\alpha}{\pi} \int_0^1 dz\, z = \frac{\alpha}{2\pi}. \tag{1.174}$$

Recalling the relation found above between F_2 and the $g - 2$ anomalous magnetic moment we find, to order $\mathcal{O}(\alpha)$,

$$g - 2 = 2F_2(q^2 = 0) \Rightarrow a_e := \frac{g - 2}{2} = F_2(0) = \frac{\alpha}{2\pi} \approx 0.0011614, \tag{1.175}$$

which may be compared to the experimental value $a_e^{exp} \approx 0.0011597$. This theoretical value was obtained by Schwinger in 1948. You should recall the words of David Gross at the 2011 Solvay conference! Read also PS, pages 196 - 198 (note the last two sentences).